

# $F$ -Signature of Affine Toric Varieties

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## 1 Introduction

The  $F$ -signature is a numerical invariant of a local (or graded) ring  $R$  containing a field of positive characteristic. It arose naturally in [SVdB97] in the study of the ring of differential operators in prime characteristic but was first singled out as an object of independent interest in [HL02]. It has been shown to be intimately connected to various measures of the singularity of  $R$ . For example, the  $F$ -signature is positive if and only if the ring is strongly  $F$ -regular, and it is equal to 1 if and only if the ring is regular [HL02]. (In general, it is a real number between 0 and 1.)

In this paper, we will compute the  $F$ -signature when  $R$  is the coordinate ring of an affine toric variety or, equivalently, a normal monomial ring. (The  $F$ -signature of a non-normal ring is zero in any case.) In particular:

**Theorem 3.3.** (cf. [WY04], Theorem 5.1) *Let  $R = k[S]$  be the coordinate ring of an affine toric variety  $X$  without torus factors, with the conventions of Remark 3.1 below. In particular,  $S = M \cap \sigma^\vee$ , where  $M$  is a lattice,  $\sigma \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$  is a full-dimensional strongly convex rational polyhedral cone, and  $\sigma^\vee$  is the dual cone to  $\sigma$ . Let  $\vec{v}_1, \dots, \vec{v}_r \in \mathbb{Z}^n$  be primitive generators for  $\sigma$ . Let  $P_\sigma \subset \sigma$  be the polytope  $\{\vec{w} \in M_{\mathbb{R}} \mid \forall i, 0 \leq \vec{w} \cdot \vec{v}_i < 1\}$ . Then  $s(R) = \text{Volume}(P_\sigma)$ .*

*More generally, suppose  $X = X' \times T$ , where  $X'$  is a toric variety without torus factors and  $T$  is an algebraic torus. Let  $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$  be the vector subspace spanned by  $\sigma$ , and let  $\sigma'$  be  $\sigma$  viewed as a cone in  $N'_{\mathbb{R}}$ . Then  $s(k[X]) = s(k[X']) = \text{Volume}(P_{\sigma'})$ .*

(We will review the notation of cones and toric varieties in the next section.) Our formula is equivalent to the one given in [WY04] in the case where  $X$  has no torus factors; when  $X$  does have torus factors, our formula corrects the one given in [WY04], which does not hold in that case.

Our method of proof differs from the proof given in [WY04]. It is similar to that used by Watanabe in [Wat99] to compute toric Hilbert-Kunz multiplicities. A different formula for the  $F$ -signature of a normal monomial ring has also been computed by Singh in [Sin05]. The methods used in this paper allow us to give an easy proof of Singh's result (Theorem 5.6).

The notion of  $F$ -signature of a pair  $(R, D)$  or a triple  $(R, D, \mathfrak{a}^t)$  has been defined in [BST11]. We compute these in the toric case:

**Theorem 4.19.** *Let  $R$  be the coordinate ring of an affine toric variety, with conventions as in Remark 3.1. Let  $D$  be a torus-invariant divisor, with associated polytope  $P_\sigma^D$  as in Definition 4.18. Then  $s(R) = \text{Volume}(P_\sigma^D)$ .*

**Theorem 4.22.** *Let  $R$  be the coordinate ring of an affine toric variety, with conventions as in Remark 3.1. Let  $D$  be a torus-invariant divisor as in Definition 4.18. Let  $\mathfrak{a} \subset R$  be a monomial ideal, with associated polytope  $P_{\sigma^{\mathfrak{a}}}^D$  as in Definition 4.21. Then  $s(R, D, \mathfrak{a}^t) = \text{Volume}(P_{\sigma^{\mathfrak{a}}}^D)$ .*

**Corollary 4.30.** *Let  $R$  be the coordinate ring of an affine toric variety,  $D$  a divisor on  $\operatorname{Spec} R$ , and  $\mathfrak{a}$  a monomial ideal, presented as in Theorem 4.22. Suppose that the pair  $(R, D)$  is  $\mathbb{Q}$ -Gorenstein. Then  $s(R, D, \mathfrak{a}^t) = \operatorname{Volume}(P_\sigma^D \cap t \cdot \operatorname{Newt}(\mathfrak{a}))$ .*

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## 2 Preliminaries

### 2.1 $F$ -signature

We recall the definition of  $F$ -signature. Let  $R$  be a ring containing a field  $k$  of characteristic  $p > 0$ . Let  $F_*^e R$  be the  $R$ -module whose underlying abelian group is  $R$  and whose  $R$ -module structure is given by Frobenius: for  $r \in R, s \in F_*^e R, r \cdot s = r^{p^e} s$ . If  $R$  is reduced, it's easy to see that  $F_*^e R$  is isomorphic to  $R^{1/p^e}$ , the  $R$ -module of  $p^e$ th roots of elements of  $R$ . (This also gives  $F_*^e R$  a natural ring structure.) Recall that  $R$  is said to be  $F$ -finite if  $F_* R$  is a finitely generated  $R$ -module. (For example, every finitely generated algebra over a perfect field is  $F$ -finite.)

*Remark 2.1.* (Conventions.) In what follows, all rings are assumed to be Noetherian  $F$ -finite domains containing a field  $k$  of prime characteristic  $p > 0$ . We assume that  $k$  is perfect unless stated otherwise (though by Remark 6.1, this assumption is mostly without loss of generality). Moreover, all rings are either local with residue field  $k$ ; graded over  $\mathbb{Z}^n$  for some  $n$  with each graded piece isomorphic to  $k$ ; or graded over  $\mathbb{N}$  with zeroeth graded piece is equal to  $k$ .

**Definition 2.2.** *Let  $R$  be a Noetherian local ring or an  $\mathbb{N}$ -graded ring with zeroeth graded piece a equal to a field. Let  $M$  be a finitely generated  $R$ -module (which is assumed to be graded if  $R$  is graded), and consider a decomposition of  $M$  as a direct sum of indecomposable  $R$ -modules. The free rank of  $M$  as an  $R$ -module is the number of copies of  $R$  in this direct sum decomposition.*

*Remark 2.3.* In general (if  $R$  is not local or  $\mathbb{N}$ -graded over a field), the free rank of a module is not well-defined. In the local or  $\mathbb{N}$ -graded setting, however, free rank is uniquely determined (see, e.g., Remark 2.6 and Lemma 6.6).

**Definition 2.4.** [HL02] *Let  $R$  be a ring (either local or graded, as described above) of dimension  $d$ . Let  $\alpha = \log_p[k^p : k] < \infty$ . For each  $e \in \mathbb{N}$ , let  $a_e$  be the free rank of  $R^{1/p^e}$  as an  $R$ -module, so that  $F_*^e R = R^{\oplus a_e} \oplus M_e$ , with no copies of  $R$  in the direct sum decomposition of  $M_e$ . We define the  $F$ -signature of  $R$  to be the limit*

$$s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e(d+\alpha)}}.$$

*Remark 2.5.* Tucker [Tuc11] showed that the limit given in Definition 2.4 exists when  $R$  is a local ring. (It follows from Lemma 6.6.5 that  $F$ -signature is well-defined when  $R$  is  $\mathbb{N}$ -graded as in Remark 2.1.)

For the sake of simplicity, we will confine ourselves to the case of perfect  $k$  at first, so that  $s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$ . After we have proved our main results in the perfect case, it will not be difficult to extend them to the case  $[k^p : k] < \infty$ .

*Remark 2.6.* The  $F$ -signature of a local ring  $(R, m, k)$  may also be characterized as follows (see [Tuc11]): define  $I_e \subset R$  to be the ideal  $\{r \in R \mid \forall \phi \in \text{Hom}_R(R^{1/p^e}, R), \phi(r^{1/p^e}) \in m\}$ . In other words,  $I_e$  is the ideal of elements of  $R$  whose  $p^e$ th roots do not generate a free summand of  $R^{1/p^e}$ . Then  $a_e = l(F_*^e(R/I_e))$ , so  $s(R) = \lim_{e \rightarrow \infty} \frac{l(F_*^e(R/I_e))}{p^{e(d+\alpha)}}$ .

Since  $l(F_*^e M) = [k^p : k]^e l(M)$ , we arrive at the following definition of  $F$ -signature, which does not depend on  $\alpha$ :

$$s(R) = \lim_{e \rightarrow \infty} \frac{l(R/I_e)}{p^{ed}}.$$

## 2.2 Affine toric varieties

Here, we present enough background on toric varieties to prove Theorem 3.3. Almost all notation is standard as in Fulton's book [Ful93], which the reader may consult for further details.

A toric variety  $X$  may be defined as a normal variety which contains an algebraic torus  $T = \text{Spec } k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$  as an open dense subset, so that the action of  $T$  on itself extends to an action of  $T$  on  $X$ . Toric varieties can be presented in terms of simple combinatorial data, making algebro-geometric computations easier on toric varieties.

Let  $N$  be a free abelian group of rank  $n$ . Let  $M = N^* = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  the dual group to  $N$ . Consider  $M$  as a lattice, called the *character lattice*, in the  $\mathbb{R}$ -vector space  $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ . Let  $k[M]$  be the semigroup ring on  $M$ , so that up to non-canonical isomorphism,  $k[M] \simeq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  is the coordinate ring of an “algebraic torus.” Elements of the semigroup  $M$  are called *characters* but may also be thought of as exponents; the inclusion of abelian groups  $\chi : M^+ \hookrightarrow (k[M])^\times$  is called the *exponential map* and is written  $m \mapsto \chi^m$ . Elements  $\chi^m \in k[M]$  are called *monomials*. In this paper, a *monomial ring*  $R$  is a  $k$ -subalgebra of  $k[M]$ , finitely generated by monomials:  $R = k[S]$ , where  $\chi^S$  is the set of monomials in  $R$ . Of course, the set of monomials in  $R$  forms a semigroup under multiplication which is naturally isomorphic to  $S$ . We denote by  $L = \text{Lattice}(S)$  the (additive) subgroup of  $M$  generated by  $S$ , which is isomorphic under the exponential map to the (multiplicative) group of monomials in  $\text{Frac}(R)$ .

In what follows, let  $\sigma \subset N_{\mathbb{R}}$  be a *strongly convex rational polyhedral cone*. By *rational polyhedral cone* we mean that  $\sigma$  is the cone of vectors  $\{\sum_i a_i \vec{v}_i \mid 0 \leq a_i \in \mathbb{R}\}$ , where  $\vec{v}_i \in M$  are a collection of finitely many *generators* for  $\sigma$ . Moreover, we require that  $\sigma$  be *strongly convex*: that is, if  $0 \neq \vec{v} \in \sigma$  then  $-\vec{v} \notin \sigma$ .

A minimal set of generators of a cone is uniquely determined up to rescaling. (For each  $i$ ,  $\mathbb{R}_{\geq 0} \cdot \vec{v}_i$  is a ray which forms one edge of the cone  $\sigma \subset N_{\mathbb{R}}$ .) It is often useful to take the vectors  $\vec{v}_i$  to be *primitive generators*: that is, we replace each  $\vec{v}_i$  with the shortest vector in  $N$  that lies on the same ray. The primitive generators of  $\sigma$  are themselves uniquely determined.

A *face* of  $\sigma$  is  $F = \sigma \cap H$ , where  $H \subset N_{\mathbb{R}}$  is a hyperplane that only intersects  $\sigma$  on its boundary  $\partial\sigma$ . (Equivalently,  $H = \vec{w}^\perp$ , where  $\vec{w} \cdot \vec{v} \geq 0$  for all  $\vec{v} \in \sigma$ . Such  $H$  is called a *supporting hyperplane*.) A codimension-one face is called a *facet*. As is (hopefully) intuitively clear, one can show that the union of the facets of  $\sigma$  is equal to the boundary of the cone,  $\partial\sigma$ . Every face of  $\sigma$  is itself a strongly convex rational polyhedral cone, whose generators are a subset of the generators of  $\sigma$ .

A strongly convex rational polyhedral cone  $\sigma$  in the vector space  $V$  has a *dual cone*,  $\sigma^\vee = \{\vec{u} \in V^* \mid \vec{u} \cdot \vec{v} \geq 0 \forall \vec{v} \in \sigma\}$ . It's a basic fact of convex geometry that  $\sigma^\vee$  is also a rational polyhedral cone, and that  $\sigma = (\sigma^\vee)^\vee$ .

Now we define affine toric varieties in the language of cones. Every affine toric variety may be presented in the following form:

**Definition 2.7.** Let  $N$  be an  $n$ -dimensional lattice,  $N \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ . Let  $M = N^*$ , and  $M_{\mathbb{R}} = M \otimes \mathbb{R}$ . Let  $\sigma \subset N_{\mathbb{R}}$  be a strongly convex rational polyhedral cone, and  $S = \sigma^{\vee} \cap M$ , where  $\sigma^{\vee} \subset M_{\mathbb{R}}$  is the dual cone to  $\sigma$ . Let  $R = k[S]$ . The affine toric variety corresponding to  $\sigma$  is defined to be  $X = \text{Spec } R$ .

*Remark 2.8.* Only normal monomial rings arise as the coordinate rings of toric varieties. Since strongly  $F$ -regular rings are normal, there will be no loss of generality in restricting our  $F$ -signature computations to only those monomial rings arising from toric varieties. (If a monomial ring does not arise in this fashion, it is not normal, hence not strongly  $F$ -regular, so we already know that its  $F$ -signature is zero.)

It will be convenient, during our  $F$ -signature computations, to temporarily assume that the cone  $\sigma$  defining our toric variety  $X_{\sigma}$  is full-dimensional. Equivalently, we assume that our toric variety contains no *torus factors*, i.e, is not the product of two lower-dimensional toric varieties, one of which is a torus. The following (easily checked) facts about products of cones will allow us to reduce to the case of a toric variety with no torus factors:

**Lemma 2.9.** ([CLS11], Proposition 3.3.9) Let  $X = \text{Spec } R$  be the affine toric variety corresponding to the cone  $\sigma$ , so that  $R = k[\sigma^{\vee} \cap M]$ . Let  $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$  be the vector subspace spanned by  $\sigma$ . Let  $N' = N'_{\mathbb{R}} \cap N$ . Let  $\sigma'$  be  $\sigma$ , viewed as a full-dimensional cone in  $N'_{\mathbb{R}}$ . Let  $N'' = N/N'$ . Then  $X \simeq X' \times T_{N''}$ , where  $X'$  is the affine toric variety (with no torus factors) corresponding to  $\sigma'$  and  $T_{N''} = \text{Spec } k[M'']$  is an algebraic torus.

On a similar note, we see that full-dimensionality is a dual property to strong convexity:

**Lemma 2.10.** ([Ful93], §1.2) A (rational polyhedral) cone is full-dimensional if and only if its dual cone  $\sigma^{\vee}$  is strongly convex. (Or, equivalently, if and only if  $\vec{0}$  is the only unit in  $\sigma^{\vee} \cap M$ .)

The following fact will also be useful later. It says that the group  $\text{Lattice}(S)$  generated by the semigroup  $S$  is equal to the character lattice  $M$ .

**Lemma 2.11.** Let  $N$  be a rank- $n$  lattice,  $M = N^*$ ,  $\sigma \subset N_{\mathbb{R}}$  a strongly convex rational polyhedral cone, and  $S = \sigma^{\vee} \cap M$  (so that  $\text{Spec } k[S]$  is the affine toric variety corresponding to  $\sigma$ ). Then  $\text{Lattice}(S) = M$ . More generally, if  $L' \subset M_{\mathbb{R}}$  is any  $n$ -dimensional lattice,  $\sigma^{\vee} \subset M_{\mathbb{R}}$  any  $n$ -dimensional cone, and  $S = \sigma^{\vee} \cap L'$ , then  $\text{Lattice}(S) = L'$ , that is,  $L'$  is the group generated by the semigroup  $S$ .

Next, we define a polytope:

**Definition 2.12.** A polytope in  $\mathbb{R}^n$  is the convex hull of a finite set of points, which we will call *extremal points*. Equivalently, a polytope is a bounded set given as the intersection of finitely many closed half-spaces  $H = \{\vec{v} \mid \vec{v} \cdot \vec{u} \geq 0\}$  (reference), or a bounded set defined by finitely many linear inequalities.

*Remark 2.13.* We will abuse notation by allowing polytopes to be intersections of half-spaces which are either open ( $H = \{\vec{v} \mid \vec{v} \cdot \vec{u} > 0\}$ ) or closed. (We will compute  $F$ -signatures to be the volumes of various polytopes. Since the volume of an intersection of half-spaces is the same whether the half-spaces are open or closed, this technicality will not affect our arguments.)

### 3 Toric $F$ -Signature Computation

*Remark 3.1.* (Conventions.) For the remainder of this paper,  $N$  is a lattice;  $M = N^*$  is the dual lattice;  $\sigma \subset N$  is a strongly convex rational polyhedral cone;  $S = M \cap \sigma^\vee$ , so that  $k[S]$  is the coordinate ring of an affine toric variety in the notation of [Ful93], and  $\vec{v}_1, \dots, \vec{v}_r$  are primitive generators for  $\sigma$ . Unless stated otherwise,  $k$  is perfect, and  $\sigma$  is full-dimensional (i.e., the associated toric variety has no torus factors).

#### 3.1 Statement of the main result, and an example

**Definition 3.2.** Let  $\sigma$  be a cone as in Remark 3.1, with primitive generators  $\vec{v}_1, \dots, \vec{v}_r$ . We define  $P_\sigma \subset \sigma^\vee$  to be the polytope  $\{\vec{w} \in M_\mathbb{R} \mid \forall i, 0 \leq \vec{w} \cdot \vec{v}_i < 1\}$ .

**Theorem 3.3.** Let  $R$  be the coordinate ring of an affine toric variety  $X$  with no torus factors, with the conventions of Remark 3.1. Then  $s(R)$  is the volume of  $P_\sigma$ . More generally, suppose  $X = X' \times T$ , where  $X'$  is a toric variety without torus factors and  $T$  is an algebraic torus. Let  $N'_\mathbb{R} \subset N_\mathbb{R}$  be the vector subspace spanned by  $\sigma$ , and let  $\sigma'$  be  $\sigma$  viewed as a cone in  $N'_\mathbb{R}$ . Then  $s(k[X]) = s(k[X']) = \text{Volume}(P_{\sigma'})$ .

We will prove this theorem in Section 3.3. For now, we provide an example computation:

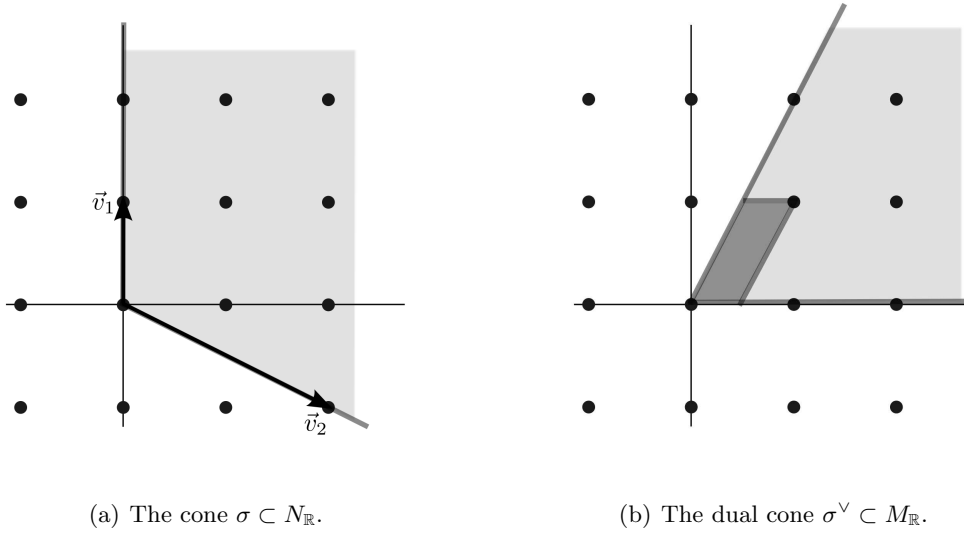


Figure 1: Computing the  $F$ -signature of the coordinate ring  $k[x, xy, xy^2]$  of a quadric cone.

*Example 3.4.* Figure 1(a) shows the cone  $\sigma$  corresponding to a plane quadric  $\mathbb{V}(xy - z^2)$ , with primitive generators  $\vec{v}_1, \vec{v}_2$ . Figure 1(b) shows the dual cone  $\sigma^\vee$ . The coordinate ring  $k[\sigma^\vee \cap M]$  is  $k[x, xy, xy^2]$ . In this case,  $P_\sigma$ , shaded in the figure, is the parallelogram  $\{(x, y) \mid 0 \leq y < 1, 0 \leq 2x - y < 1\}$ . The  $F$ -signature is  $s(R) = \text{Volume}(P_\sigma) = \frac{1}{2}$ .

#### 3.2 $R$ -module Decomposition of $R^{1/q}$

The main supporting result proved in this section is Lemma 3.8, which gives a formula for the free rank of  $R^{1/q}$  as an  $R$ -module in terms of the number of monomials in  $R^{1/q}$  having a certain property. That lemma will be integral to our proof of the main theorem. Lemma 3.8

follows immediately from Lemma 3.7, which describes how  $R^{1/q}$  decomposes as a direct sum of indecomposable  $R$ -module.

We will be able to compute the  $F$ -signature of a monomial ring  $R$  because the  $R$ -module  $R^{1/q}$  has an especially nice graded structure. In particular:

**Lemma 3.5.** *Let  $R$  be a monomial ring, with  $q = p^e$ , and with character lattice  $M \simeq \mathbb{Z}^n$ . Then:*

1.  $R^{1/q}$  is finitely generated, as an  $R$ -module, by  $q^{\text{th}}$  roots of monomials in  $R$  of bounded degree.
2.  $R^{1/q}$  admits a natural  $\frac{1}{q}M$ -grading which respects the  $M$ -grading on  $R \subset R^{1/q}$ . Each graded piece of  $R^{1/q}$  is a one-dimensional  $k$ -vector space.

*Proof.* 1. If  $R$  is generated by a finite set of monomials  $\tau_i$ , we can pick a minimal set of  $t$  generators from among  $\{\prod_i \tau_i^{a_i/q} \mid 0 \leq a_i < q\}$ .

2. We consider  $R$  as a graded subring of the  $M$ -graded ring  $k[M]$ . The grading on  $R^{1/q}$  is inherited in the obvious way:  $\deg(\chi^m)^{1/q} = \frac{1}{q} \deg \chi^m = \frac{1}{q}m$ . We conclude that  $R^{1/q}$  has a natural  $\frac{1}{q}M$ -grading. Each graded piece consists of the set of  $k$ -multiples of a single monomial  $\chi^{m/q}$ .

□

It's well-known that relations on graded modules over monomial rings are generated by so-called “binomial” relations. We supply a proof here, for lack of a better reference:

**Lemma 3.6.** *Let  $W$  be a  $G$ -graded  $R$ -module,  $G$  an abelian group, with each nonzero graded piece a one-dimensional  $k$ -vector space. (For example, when  $R$  is a monomial subring of  $k[x_1, \dots, x_n]$ ,  $W = R^{1/q}$  is  $(\mathbb{Z}/q)^n$ -graded.)*

1. We can write  $W$  as a quotient of a free module so that the relations are generated by “binomial” relations, of the form  $r \cdot \rho = s \cdot \mu$ , for  $r, s$  homogeneous elements of  $R$  and  $\rho, \mu$  homogeneous elements of  $W$  such that  $\deg r + \deg \rho = \deg s + \deg \mu$ .
2. We will say that two monomials  $\rho, \mu \in W$  are related if they satisfy a binomial relation. Then being related is an equivalence relation.

*Proof.* 1. Let  $\mu_i$  be graded generators for  $W$ . Let  $\sum_i r_i \mu_i = 0$  be a relation. Since  $W$  is graded,  $\sum_i r_i \mu_i$  may be written as a sum of graded pieces, each of which is itself equal to 0. In other words, the relations on  $W$  are generated by relations with the property that  $r_i \mu_i$  has the same degree for each  $i$ . In that case, since each graded piece of  $W$  is a one-dimensional  $k$ -vector space, we have that for each  $i$ , and each  $j$  for which  $r_i \mu_i \neq 0$ ,  $r_i \mu_i = c_{ij} r_j \mu_j$  for some  $c_{ij} \in k$ . This is a binomial relation on  $\mu_i$  and  $\mu_j$ , and binomial relations of this form generate the original relation  $\sum_i r_i \mu_i = 0$ .

2. If  $r\rho = s\mu$ , and  $s'\mu = t\tau$ , then  $rs'\rho = st\tau$ , so  $\rho \sim \tau$ .

□

The following lemma essentially indicates how to decompose  $R^{1/q}$  as a direct sum of  $R$ -submodules generated by monomials. It also gives a condition describing which monomials generate free summands of  $R^{1/q}$ .

**Lemma 3.7.** *Let  $W$  be a finitely generated  $G$ -graded  $R$ -module,  $G$  an abelian group, with each nonzero graded piece a one-dimensional  $k$ -vector space. (For example,  $W = R^{1/q}$ ,  $G = \mathbb{Z}^n$ .) Let  $H$  be a set of homogeneous generators for  $W$ . Let  $A_1, \dots, A_k \subset H$  be the distinct equivalence classes of elements of  $H$  which are related (in the sense of Lemma 3.6). Then:*

1.  $W \simeq \bigoplus_i R \cdot A_i$  is a direct sum of submodules generated by the sets  $A_i$ .
2. Each of the submodules  $R \cdot A_i$  is rank one (hence indecomposable even as an ungraded  $R$ -module).
3. Finally, a homogeneous element  $\mu \in W$  generates a free summand of  $W$  if and only if the only homogeneous elements of  $W$  that are related to  $\mu$  are  $R$ -multiples of  $\mu$ .

*Proof.* 1. Suppose that  $A, B \subset H$ , and  $C = A \amalg B$ . Since the corresponding submodules  $R \cdot A$  and  $R \cdot B$  are graded, their intersection must also be graded. In particular, in order for these modules to have nonempty intersection (equivalently, for the sum  $R \cdot A + R \cdot B = R \cdot C$  to fail to be direct), we should have a binomial relation  $r\mu = s\tau$  for some  $\mu \in A, \tau \in B$ , and  $r, s \in R$ , by Lemma 3.6. However, we constructed the sets  $A_i$  so that no binomial relations exist between them. We conclude that the sum is direct, and  $W = R \cdot H \simeq \sum_i R \cdot A_i$ .

2. Suppose that we have a subset  $A \subset H$  of homogeneous elements which are all related to one another (for example,  $A = A_i$  for some  $i$ ). Then pick a homogeneous  $\mu \in A$ . For any  $\tau \in A$ , we have that  $r\mu = s\tau$  for  $r, s \in R$ ; equivalently,  $\tau = \frac{r}{s}\mu \in \text{Frac}(R) \cdot \mu$ . We conclude that  $R \cdot A$  has rank 1, that is,  $R \cdot A \otimes_R \text{Frac}(R) \simeq \text{Frac}(R)$ .
3. Fix  $\mu \in W$ . Let  $A_1, \dots, A_k$  be the equivalence classes of related monomials in  $W$ . Without loss of generality, we may assume that  $\mu \in A_1$ . Then  $R \cdot A_1$  is free of rank one if and only if it is generated by a single monomial. Thus,  $R \cdot \mu$  is a free summand of  $W$  if and only if  $\mu$  generates  $R \cdot A_1$ , that is, if and only if  $\mu$  divides every homogeneous element of  $W$  that is related to  $\mu$ .

□

The following lemma will be essential in the next section when we compute the free rank  $a_e$  of  $R^{1/q}$  as an  $R$ -module. (We will also use this lemma in Section 4 when computing the  $F$ -signature of pairs and triples.)

**Lemma 3.8.** *Let  $R = k[S]$  be a monomial ring,  $S$  a semigroup, and let  $L = \text{Lattice}(S)$  be the group generated by  $S$ . Fix  $q = p^e$ . Let  $H \subset \frac{1}{q}L$  be a finitely generated  $S$ -module, so that  $k[H] \subset k[\frac{1}{q}L]$  is an  $R$ -module finitely generated by monomials. Let  $a_e$  be the free rank of  $k[H]$  as an  $R$ -module. Then the set of monomials in  $H$  which generate a free summand of  $k[H]$  is  $\{\chi^{\vec{v}} \mid \vec{v} \in H, \text{ and } \forall \vec{k} \in L \setminus S, \vec{v} + \vec{k} \notin H\}$ . Moreover, if 0 is the only unit in  $H$ , then  $a_e$  is the size of this generating set.*

*Proof.* By Lemma 3.7, a monomial  $\mu \in k[H]$  generates a free summand of  $k[H]$  if and only if it is unrelated to all monomials in  $k[H]$  that are not  $R$ -multiples of itself. We may characterize  $\tau$  being related to  $\mu$  (but not a multiple of it) by  $\tau = \frac{r}{s}\mu$ , with  $\frac{r}{s} \in (\text{Frac } R) \setminus R$ . Thus, the set of monomial generators for  $k[H]$  which generate a free summand of  $k[H]$  is  $\{\mu \in \chi^H \mid \text{for all monomials } \frac{r}{s} \in (\text{Frac } R) \setminus R, \frac{r}{s}\mu \notin k[H]\}$ . We may rewrite this set as  $\{\chi^{\vec{v}} \mid \vec{v} \in H, \text{ and } \forall \vec{k} \in L \setminus S, \vec{v} + \vec{k} \notin H\}$ , which is precisely the set described in the statement of the lemma. If 0 is the only unit in  $H$ , then there is a one-to-one-correspondence between monomials in the generating set and free summands of  $W$ . In that case,  $a_e$  is the size of the generating set. □

*Remark 3.9.* As we'll see shortly, when we apply Lemma 3.8 to the case of the  $R$ -module  $R^{1/q}$ , the technical requirement that  $0 \in H$  be the only unit corresponds to the cone  $\sigma$  being full-dimensional.

### 3.3 Proof of the Main Result

*Remark 3.10.* (An aside on computing volumes.) Consider  $M \subset M_{\mathbb{R}}$ , a lattice abstractly isomorphic to  $\mathbb{Z}^n$  contained in a vector space abstractly isomorphic to  $\mathbb{R}^n$ . Choosing a basis for  $M$  gives us an identification of  $M_{\mathbb{R}}$  with  $\mathbb{R}^n$ , hence a way to measure volume on  $M_{\mathbb{R}}$ . It's easily checked that this volume measure depends only on  $M$  and not on our choice of basis for  $M$ . (Such a measure is uniquely determined by the fact that with respect to it, the measure of a fundamental parallelepiped for  $M$ , also called the *covolume* of  $M$ , is 1.) Thus, it makes sense to talk about measuring volume “relative to the lattice  $M$ ,” denoted  $\text{Volume}_M$  (or simply  $\text{Volume}$  when there is no risk of ambiguity).

Now we are ready to prove our main result.

*Proof of Theorem 3.3.* Suppose first that  $X$  has no torus factors. We apply Lemma 3.8, with  $H = \sigma^\vee \cap \frac{1}{p^e}M$ ,  $k[H] = R^{1/p^e}$ . Since  $\sigma$  is full-dimensional,  $\sigma^\vee$  is strongly convex (by Lemma 2.10), so  $H$  contains no nontrivial units. Then

$$a_e = \#\{\vec{v} \in (\sigma^\vee \cap \frac{1}{p^e}M) \mid \forall \vec{k} \in M \setminus \sigma^\vee, \vec{v} + \vec{k} \notin \sigma^\vee\}.$$

Let  $P'_\sigma$  be the set  $\{\vec{v} \in \sigma^\vee \mid \forall \vec{k} \in M \setminus \sigma^\vee, \vec{v} + \vec{k} \notin \sigma^\vee\}$ . Then  $a_e = \#\{\vec{v} \in P'_\sigma \cap \frac{1}{p^e}M\}$ . By Lemma 3.11,  $P'_\sigma = P_\sigma$ . Set  $q = p^e$ . Then  $s(R)$ , defined to be  $\lim_{e \rightarrow \infty} \frac{a_e}{p^{en}}$ , is equal to  $\lim_{q \rightarrow \infty} \frac{\#(P_\sigma \cap \frac{1}{q}M)}{q^n}$ . We apply Lemma 3.12 to conclude that  $s(R) = \text{Volume}(P_\sigma)$ .

Suppose now that  $X$  has torus factors. By Lemma 2.9,  $X \simeq X' \times T_{N''}$ , where  $X' = \text{Spec } k[\sigma' \cap M']$  and  $T_{N''}$  is the algebraic torus  $\text{Spec } k[M'']$ . In particular,  $R \simeq k[X'] \otimes_k k[M'']$ . We apply Theorem 6.2 on the  $F$ -signature of products to see that  $s(R) = s(k[X']) \cdot 1 = s(k[X']) = \text{Volume}(P_{\sigma'})$ . (It is easy to check directly that  $s(k[M'']) = 1$ : writing  $M'' \simeq \mathbb{Z}^{d''}$ , we see that  $k[\mathbb{Z}^{d''}]^{1/q}$  is a free  $k[\mathbb{Z}^{d''}]$ -module of rank  $q^{d''}$ .)  $\square$

It remains to prove the two lemmas referenced in the proof of Theorem 3.3.

**Lemma 3.11.** *Suppose that we are in the situation of Remark 3.1. Then*

$$P'_\sigma := \{\vec{v} \in \sigma^\vee \mid \forall \vec{k} \in M \setminus \sigma^\vee, \vec{v} + \vec{k} \notin \sigma^\vee\} = \{\vec{v} \in M_{\mathbb{R}} \mid \forall i, 0 \leq \vec{v} \cdot \vec{v}_i < 1\} =: P_\sigma.$$

*Proof.* Recall that  $\sigma^\vee = \{\vec{u} \mid \vec{u} \cdot \vec{v}_i \geq 0 \text{ for all } i\}$ .

Suppose  $\vec{v} \in P_\sigma$ , so that for each  $i$ ,  $0 \leq \vec{v} \cdot \vec{v}_i < 1$ . Fix  $\vec{k} \in M \setminus \sigma^\vee$ . Since  $\vec{k} \notin \sigma^\vee$ , we know that  $\vec{k} \cdot \vec{v}_j < 0$  for some  $j$ . For such  $j$ , since  $\vec{k} \cdot \vec{v}_j \in \mathbb{Z}$ , we know that  $\vec{k} \cdot \vec{v}_j \leq -1$ . It follows that  $(\vec{v} + \vec{k}) \cdot \vec{v}_j < 0$ . Thus,  $\vec{v} + \vec{k} \notin \sigma^\vee$ . We conclude that  $\vec{v} \in P'_\sigma$ . Hence,  $P_\sigma \subset P'_\sigma$ .

Conversely, suppose that  $\vec{v} \notin P_\sigma$ , so that for some  $j$ ,  $\vec{v} \cdot \vec{v}_j \geq 1$ . Set  $\vec{k}_0$  to be any vector in  $M$  such that  $\vec{k}_0 \cdot \vec{v}_j = -1$ . (Such  $\vec{k}_0$  exists since by Lemma 2.11,  $\text{Lattice}(S) = M$ .) Choose  $\vec{k}_1$  to be any vector in  $M$  that also lies in the interior of the facet  $F_j = \vec{v}_j^\perp \cap \sigma^\vee$  of  $\sigma^\vee$ . Then  $\vec{k}_1 \cdot \vec{v}_i > 0$  for each  $i \neq j$ . Thus, for sufficiently large  $m$ ,  $(\vec{k}_0 + m\vec{k}_1) \cdot \vec{v}_i \geq 0$  for  $i \neq j$ , while  $(\vec{k}_0 + m\vec{k}_1) \cdot \vec{v}_j = 0$ . Set  $\vec{k} = \vec{k}_0 + m\vec{k}_1$ . Then  $\vec{k} \in M$ , but  $\vec{v} \notin \sigma^\vee$ , since  $\vec{k} \cdot \vec{v}_j = -1 < 0$ . On the other hand,  $\vec{v} + \vec{k} \in \sigma^\vee$ , since  $(\vec{v} + \vec{k}) \cdot \vec{v}_i \geq 0$  for each  $i$ . We conclude that  $\vec{v} \notin P'_\sigma$ .

Hence,  $P'_\sigma \subset P_\sigma$ . We conclude that  $P_\sigma = P'_\sigma$ , as we desired to show.  $\square$



**Lemma 3.12.** *Let  $M$  be a lattice and  $P \subset M \otimes_{\mathbb{Z}} \mathbb{R}$  a polytope (or, more generally, any set whose boundary has measure zero). Then  $\lim_{q \rightarrow \infty} \frac{\#\{P \cap \frac{1}{q}M\}}{q^n} = \text{Volume}(P)$ .*

*Proof.* In fact, when  $P$  is a polytope, it can be shown that the quantity  $\#\{P \cap \frac{1}{q}M\}$  is polynomial in  $q$  of degree  $n$ , known as the *Ehrhart polynomial* of  $P$ , and that its leading coefficient  $\lim_{q \rightarrow \infty} \frac{\#\{P \cap \frac{1}{q}M\}}{q^n}$  is  $\text{Volume}(P)$  ([MS05], Thm 12.2). Even without this fact, however, it is easy to sketch a proof of the special case that we require: the quantity  $\lim_{q \rightarrow \infty} \frac{\#\{P \cap \frac{1}{q}M\}}{q^n}$  is a limit of Riemann sums measuring the volume of  $P$  with respect to the lattice  $M$ . (See, for example, [Fol99], Theorem 2.28.)  $\square$

## 4 $F$ -Signature of Pairs and Triples

### 4.1 Definitions

The notion of singularities of pairs is an important one in birational geometry. Instead of studying the singularities of a variety  $X$ , or the singularities of a divisor  $D$  on  $X$ , one studies the pair  $(X, D)$ , for example by considering how  $D$  changes under various resolutions of  $X$ . (See [Kol97] for an introduction to pairs in this setting.) More generally, it is often useful to study triples  $(X, D, \mathfrak{a})$ , where  $D$  is a divisor and  $\mathfrak{a}$  an ideal sheaf on  $X$ .

The  $F$ -signature of pairs and triples was recently defined in [BST11]. First, we recall:

**Definition 4.1.** *Let  $R$  be a normal domain. Let  $D$  be an effective Weil divisor on  $X = \text{Spec } R$ . We define  $R(D)$  to be the module of global sections of  $\mathcal{O}_X(D)$ . That is,  $R(D) = \{f \in \text{Frac}(R) \mid \text{div } f + D \geq 0\}$ .*

*Remark 4.2.* Note that  $R \subset R(D)$ , and that if  $D$  is the principal divisor  $\text{div } g$  for some  $g \in R$ , then  $R(D) = R \cdot \frac{1}{g}$  is the cyclic  $R$ -module generated by  $\frac{1}{g}$ .

Now we can define the  $F$ -signature of a pair or triple:

**Definition 4.3.** *Let  $(R, m)$  be a normal local (or  $\mathbb{N}$ -graded) domain over  $k$ , of dimension  $d$ . Let  $D = \sum_i a_i D_i$  be an effective  $\mathbb{Q}$ -divisor on  $X = \text{Spec } R$ ,  $\mathfrak{a}$  an ideal of  $R$ , and  $0 \leq t \in \mathbb{R}$ . We define the  $F$ -signature of the triple  $(R, D, \mathfrak{a}^t)$  as follows. For each  $e$ , let  $\mathfrak{C}_e = \text{Hom}_R(R^{1/p^e}, R)$ . Let  $\mathfrak{D}_e$  be the  $R^{1/p^e}$ -submodule of  $\mathfrak{C}_e$  defined as  $\text{Hom}_R(R(\lceil (p^e - 1)D \rceil)^{1/p^e}, R)$ . (The module structure is given by premultiplication,  $r^{1/p^e} \cdot \phi(x) = \phi(r^{1/p^e} x)$ .) Define  $I_e^D \subset R$  to be the ideal  $\{r \in R \mid \forall \phi \in \mathfrak{D}_e, \phi(r^{1/p^e}) \in m\}$ . Then the  $F$ -signature  $s(R, D)$  is  $\lim_{e \rightarrow \infty} \frac{l(R/I_e^D)}{p^{ed}}$ .*

*Let  $\mathfrak{D}'_e = \mathfrak{D}_e \cdot (F_*^e(\mathfrak{a}^{\lceil (p^e - 1)t \rceil}))$ , the  $R^{1/p^e}$ -submodule of  $\mathfrak{D}_e$  generated by  $\{[x \mapsto \phi(\mathfrak{a}^{1/p^e} \cdot x)] \mid \mathfrak{a}^{\lceil (p^e - 1)t \rceil}, \phi \in \mathfrak{D}_e\}$ .*

*Define  $I_e^{\mathfrak{a}} \subset R$  to be the ideal  $\{r \in R \mid \forall \phi \in \mathfrak{D}'_e, \phi(r^{1/p^e}) \in m\}$ . The  $F$ -signature  $s(R, D, \mathfrak{a}^t)$  is defined to be  $\lim_{e \rightarrow \infty} \frac{l(R/I_e^{\mathfrak{a}})}{p^{ed}}$ .*

*Remark 4.4.* The limits given in Definition 4.3 have been shown to exist in [BST11], in the case of a local ring.

*Remark 4.5.* It's easily checked that the  $F$ -signature of the triple  $(R, D, (1))$  (with  $\mathfrak{a}$  as the unit ideal) is the  $F$ -signature of the pair  $(R, D)$ . Likewise, the  $F$ -signature of the pair  $(R, 0)$  (with  $D$  as the zero divisor) is the  $F$ -signature of  $R$ .

Just like the “usual”  $F$ -signature, the  $F$ -signature of pairs may be viewed as a measure of the number of splittings of the Frobenius map, or as a measure of the number of free summands splitting off from  $R^{1/p^e}$ , though the  $F$ -signature of pairs only counts certain summands:

**Lemma 4.6.** (See [BST11], Proposition 3.5.) Suppose that we are in the setting of Definition 4.3. Set  $a_e^D = l(F_*(R/I_e^D))$ , so that  $s(R, D) = \lim_{e \rightarrow \infty} \frac{a_e^D}{p^{e(d+\alpha)}}$ . Then  $a_e^D$  is the maximum rank of a free summand of  $R/[(p^e - 1)D]^{1/p^e}$  that is simultaneously a free summand of the submodule  $R^{1/p^e}$ . Moreover, any  $k$ -vector space basis for  $R^{1/p^e}/(I_e^D)^{1/p^e}$  lifts to a set of generators in  $R^{1/p^e}$  for such a free summand of maximum rank, and  $(I_e^D)^{1/p^e}$  is the submodule of elements of  $R^{1/p^e}$  which do not generate such a free summand.

Minor modifications can be made to  $\mathfrak{D}_e$  or to  $\mathfrak{D}'_e$  in the definition, without changing the  $F$ -signature. In particular:

**Lemma 4.7.** ([BST11], Lemma 4.17.) Let  $R, D, \mathfrak{a}, t$  be as in Definition 4.3. Suppose that for each  $e$ ,  $\mathfrak{D}_e$  is replaced by some  $\mathfrak{D}'_e \subset \mathfrak{C}_e$  such that for some  $0 \neq c \in R$ ,  $c^{1/p^e} \mathfrak{D}_e \subset \mathfrak{D}'_e$  and  $c^{1/p^e} \mathfrak{D}'_e \subset \mathfrak{D}_e$ . Then the limits given in Definition 4.3 are unchanged by this replacement.

This lemma allows us to make many simplifications in our  $F$ -signature computations:

**Lemma 4.8.** ([BST11], discussion following Lemma 4.17.) Let  $R, D, \mathfrak{a}, t$  be as in Definition 4.3.

Suppose that for some sequence of divisors  $D'_e$ , the coefficients of  $(p^e - 1)D - D'_e$  are bounded. Then replacing  $\mathfrak{D}_e = \text{Hom}_R(R/[(p^e - 1)D]^{1/p^e}, R)$  with  $\text{Hom}_R(R/[D'_e]^{1/p^e}, R)$  in Definition 4.3 does not change  $s(R, D, \mathfrak{a}^t)$ . (In particular, we may replace  $(p^e - 1)D$  by  $p^e D$  in the limits given in Definition 4.3 without changing the  $F$ -signature.) If we write  $D = \sum_i a_i D_i$ , then  $s(R, D, \mathfrak{a}^t)$  is continuous in  $a_i$ .

Likewise, suppose  $[(p^e - 1)t] - t'_e$  is bounded for some sequence of exponents  $t'_e$ . Then replacing  $[(p^e - 1)t]$  with  $[t'_e]$  in the definition does not change  $s(R, D, \mathfrak{a}^t)$ . Also,  $s(R, D, \mathfrak{a}^t)$  is continuous in  $t$ .

Finally, replacing  $\mathfrak{a}^{t'_e}$  with its integral closure  $\overline{\mathfrak{a}^{t'_e}}$  does not change the  $F$ -signature.

Lemma 4.8 will allow us to simplify our computations later by, for example, assuming that  $t$  and the coefficients  $a_i$  of  $D$  are rational numbers with denominator a power of  $p$ .

We also have, in the triples case:

**Lemma 4.9.** Suppose that we are in the setting of Definition 4.3. Then  $I_e^{\mathfrak{a}} = (I_e^D : \mathfrak{a}^{\lceil t(p^e - 1) \rceil})$ . Equivalently, set  $a_e^{\mathfrak{a}} = l(F_*^e R / F_*^e I_e^{\mathfrak{a}})$ , so that  $s(R, D, \mathfrak{a}^t) = \lim_{e \rightarrow \infty} \frac{a_e^{\mathfrak{a}}}{p^{e(d+\alpha)}}$ . Then  $I_e^{\mathfrak{a}} = (I_e^D :_R \mathfrak{a}^{\lceil t(p^e - 1) \rceil})$ , and

$$a_e^{\mathfrak{a}} = l(F_*^e R / (F_*^e I_e^D :_{R^{1/p^e}} F_*^e \mathfrak{a}^{\lceil t(p^e - 1) \rceil})).$$

*Proof.* Requiring that  $r \in (I_e^D : \mathfrak{a}^{\lceil t(p^e - 1) \rceil})$  is the same as requiring that multiplication by an element  $a \in (\mathfrak{a}^{\lceil t(p^e - 1) \rceil})^{1/p^e}$  sends  $r^{1/p^e}$  into  $(I_e^D)^{1/p^e}$ , so that for all  $\phi \in \mathfrak{D}_e$ ,  $\phi(a \cdot r^{1/p^e}) \in m$ . This is equivalent to saying that for all  $\phi \in \mathfrak{D}'_e$ ,  $\phi(r^{1/p^e}) \in m$ . We conclude that  $r \in (I_e^D : \mathfrak{a}^{\lceil t(p^e - 1) \rceil}) \iff r \in I_e^{\mathfrak{a}}$ . We have proved our first claim; the second claim follows immediately.  $\square$

## 4.2 Toric preliminaries for pairs and triples

For our pair and triple computations, we will require some understanding of divisors on toric varieties. (Unless stated otherwise, proofs of these results may be found in [Ful93]).

**Definition 4.10.** A prime Weil divisor  $D$  on a toric variety  $X$  is torus-invariant if it is invariant under the action of the embedded torus on  $X$ . More generally, a divisor  $D$  is torus-invariant if  $D = \sum_i a_i D_i$ , where  $D_i$  are the torus-invariant prime divisors of  $X$ .

The torus-invariant prime divisors of  $X$  are in bijective correspondence with primitive generators  $\vec{v}_i$  of the cone  $\sigma \subset N_{\mathbb{R}}$  of  $X$ . (In particular, the prime divisor corresponding to  $\vec{v}_i$  is  $D_i = \mathbb{V}(I_i)$ , where  $I_i$  is the ideal generated by monomials  $\vec{u}$  such that  $\vec{u} \cdot \vec{v}_i \neq 0$ .) It can be shown that  $\nu_i : \text{Frac}(R) \rightarrow \mathbb{Z}$ , the discrete valuation corresponding to  $D_i$ , is given by  $\nu_i(\vec{u}) = \vec{u} \cdot \vec{v}_i$ . From this, it follows that:

**Lemma 4.11.** *Let  $X = \text{Spec } R$  be an affine toric variety,  $R = k[S]$ ,  $S = \sigma^\vee \cap M \subset M_{\mathbb{R}}$ . Let  $D = \sum_i a_i D_i$  be a torus-invariant divisor on  $X$ , where each  $D_i$  corresponds to a primitive generator  $\vec{v}_i$  of  $\sigma$ . Then  $R(D) = \sum_{\vec{u}} R \cdot x^{\vec{u}}$ , where the sum is taken over all  $\vec{u} \in S$  such that  $\vec{u} \cdot \vec{v}_i \geq -a_i$ .*

For our  $F$ -signature of triples computation, we will require the concept of the Newton polyhedron of a monomial ideal.

**Definition 4.12.** *A polyhedron is a possibly unbounded intersection of finitely many half-spaces.*

**Definition 4.13.** *Let  $R = k[S]$  be a monomial ring, with  $S \subset M_{\mathbb{R}}$ , as above. Let  $\mathfrak{a} \subset R$  be a monomial ideal (i.e., an ideal generated by monomials). The Newton polyhedron of  $\mathfrak{a}$ , a polyhedron in  $M_{\mathbb{R}}$ , is the convex hull of the set of monomials in  $\mathfrak{a}$ .*

The Newton polyhedron is closely related to the integral closure of monomial ideals:

**Lemma 4.14.** *([Vil01], Proposition 7.3.4) Let  $R = k[S]$  be a monomial ring as above. Let  $\mathfrak{a} \subset R$  be a monomial ideal. Then the integral closure  $\overline{\mathfrak{a}}$  of  $\mathfrak{a}$  in  $R$  is a monomial ideal generated by those monomials in the set  $\text{Newt}(\mathfrak{a}) \cap M$ .*

**Definition 4.15.** *Let  $Q_1, Q_2$  be subsets of  $\mathbb{R}^n$ . The Minkowski sum of  $Q_1$  and  $Q_2$ , denoted  $Q_1 + Q_2$ , is the set  $\{\vec{u}_1 + \vec{u}_2 \mid \vec{u}_1 \in Q_1, \vec{u}_2 \in Q_2\}$ . We denote by  $Q_1 - Q_2$  the set  $Q_1 + (-Q_2) = \{\vec{u}_1 - \vec{u}_2 \mid \vec{u}_1 \in Q_1, \vec{u}_2 \in Q_2\}$ .*

It's easy to see that the Minkowski sum of two polyhedrons is itself a polyhedron, and that the sum of two polytopes is a polytope. (See, for example, [Grü03], §15.1.)

As Corollary 4.30 is a statement about  $\mathbb{Q}$ -Gorenstein pairs, we recall the definition of the  $\mathbb{Q}$ -Gorenstein condition.

**Definition 4.16.** *If  $D$  is an effective divisor on  $X = \text{Spec } R$ , the pair  $(R, D)$  is  $\mathbb{Q}$ -Gorenstein if, fixing a canonical divisor  $K_X$  on  $X$ , the divisor  $K_X + D$  is  $\mathbb{Q}$ -Cartier; that is, some integer multiple of  $K_X + D$  is Cartier.*

It happens that on a toric variety, a canonical divisor may be given by  $K_X = -\sum_i D_i$ , where the sum is taken over all torus-invariant divisors on  $X$ . It's also a fact that Cartier divisors on an affine toric variety are principal. We conclude:

**Lemma 4.17.** *Let  $X = \text{Spec } k[S]$  be an affine toric variety with a corresponding strongly convex polyhedral cone  $\sigma$ . Let  $D = \sum_i a_i D_i$  be a divisor. Then  $(R, D)$  is  $\mathbb{Q}$ -Gorenstein if and only if for some  $\vec{w} \in M \otimes \mathbb{Q}$ ,  $\vec{w} \cdot \vec{v}_i = -1 + a_i$  for each  $i$ .*

*Proof.* Given  $\vec{u} \in M$ ,  $\text{div } x^{\vec{u}} = \sum_i (\vec{u} \cdot \vec{v}_i) D_i$ . This operation extends linearly to  $\mathbb{Q}$ -divisors, so that for  $\vec{u} \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ ,  $\text{div } x^{n\vec{u}} = n(\sum_i c_i D_i)$  if and only if  $\vec{u} \cdot \vec{v}_i = c_i$  for each  $i$ . Thus,  $K_X + D$  is  $\mathbb{Q}$ -Gorenstein if and only if for some  $\vec{w} \in M \otimes \mathbb{Q}$ ,  $\vec{w} \cdot \vec{v}_i = -1 + a_i$  for each  $i$ .  $\square$

### 4.3 Pairs Computation

Now we will compute the  $F$ -signature of pairs and triples. We begin with the pairs case, in which our proof requires little modification from that of Theorem 3.3.

**Definition 4.18.** Let  $\sigma$  be a cone as in Remark 3.1, with primitive generators  $\vec{v}_1, \dots, \vec{v}_r$ . Let  $D = \sum_i a_i D_i$  be a torus-invariant  $\mathbb{Q}$ -divisor on  $\text{Spec } R$ , with  $D_i$  the prime divisor corresponding to  $\vec{v}_i$ . We define  $P_\sigma^D \subset \sigma^\vee$  to be the polytope  $\{\vec{v} \in M_\mathbb{R} \mid \forall i, 0 \leq \vec{v} \cdot \vec{v}_i < 1 - a_i\}$ .

**Theorem 4.19.** Let  $R$  be the coordinate ring of an affine toric variety, with conventions as in Remark 3.1. Let  $D$  be a torus-invariant  $\mathbb{Q}$ -divisor, with associated polytope  $P_\sigma^D$  as in Definition 4.18. Then  $F_*^e I_e^D$  is generated by the monomials in the set  $(\sigma \setminus P_\sigma^D) \cap \frac{1}{p^e} M$ , and  $s(R) = \text{Volume}(P_\sigma^D)$ .

*Proof.* First, we apply Lemma 4.8 to replace  $(p^e - 1)D$  by  $p^e D$  without changing the  $F$ -signature. By the same lemma,  $s(R, D)$  is continuous as a function of the  $a_i$ , so we may assume that  $a_i \in \frac{1}{p^e} \mathbb{Z}$ . (Proving the claim on that dense subset will prove it for all divisors by continuity.) We also assume that  $e$  is sufficiently large, so that  $p^e D$  is an integral divisor, and  $\lceil p^e D \rceil = p^e D$ . As a result of all this simplification, we may ignore the rounding-up operation.

By Lemma 4.11,  $R(p^e D)$  is an  $\mathbb{N}^n$ -graded  $R$ -module, generated by  $\{\chi^{\vec{v}} \mid \vec{v} \in M, \text{ and } \vec{v} \cdot \vec{v}_i \geq -p^e a_i\}$ . It follows that  $R(p^e D)^{1/p^e}$  is  $\mathbb{N}^n/q$ -graded: it's generated by  $\{\chi^{\vec{v}} \mid \vec{v} \in \frac{1}{p^e} M, \text{ and } \vec{v} \cdot \vec{v}_i \geq -a_i\}$ .

Thus, we may apply Lemma 3.7. The graded module  $R(p^e D)^{1/p^e}$  decomposes as a direct sum of graded submodules, where each submodule is generated by related monomials. Each submodule splits off from  $R(p^e D)^{1/p^e}$  if and only if it is generated by a single monomial; likewise, each submodule generated by monomials in  $R^{1/p^e}$  splits off from  $R^{1/p^e}$  if and only if it is generated by a single monomial.

Set  $\sigma' = \{\vec{v} \in M_\mathbb{R} \mid \forall i, \vec{v} \cdot \vec{v}_i \geq -a_i\}$ , so that if  $S' := M \cap \sigma'$ , then  $\chi^{S'}$  is the set of generators for  $R(D)$ . By Lemma 3.8,  $a_e = \#\{\vec{v} \in \frac{1}{p^e} S \mid \forall \vec{k} \in M \setminus S, \vec{v} + \vec{k} \notin \frac{1}{p^e} S'\}$ . Note that  $F_*^e I_e^D$  is generated by those monomials in  $R^{1/p^e}$  whose corresponding characters are *not* in this set.

Following the proof of Theorem 3.3, we find that  $a_e = \#\{\vec{v} \in \frac{1}{p^e} S \mid \forall \vec{k} \in M \setminus S, \vec{v} + \vec{k} \notin \frac{1}{p^e} S'\}$ . Equivalently,  $a_e = \#\{\vec{v} \in \sigma^\vee \cap \frac{1}{p^e} M \mid \forall \vec{k} \in M \setminus \sigma^\vee, \vec{v} + \vec{k} \notin \sigma'\}$ . That is,

$$a_e = \#\left\{\frac{1}{p^e} M \cap P'\right\},$$

where  $P' = \{\vec{v} \mid \vec{v} \cdot \vec{v}_i \geq 0, \text{ and } \forall \vec{k} \in M \setminus \sigma^\vee, (\vec{v} + \vec{k}) \cdot \vec{v}_i < -a_i \text{ for some } i\}$ . (By the same argument,  $F_*^e I_e$  is generated by the monomials whose characters lie in  $\sigma^\vee \setminus P'$ .) By Lemma 4.20 (the pairs analogue to Lemma 3.11),  $P' = P_\sigma^D$ . Our claim then follows from our lemma on volumes of polytopes, Lemma 3.12, just as in our original proof of Theorem 3.3.  $\square$

**Lemma 4.20.** Suppose that we are in the situation of Lemma 4.19, and that  $P' = \{\vec{v} \mid \vec{v} \cdot \vec{v}_i \geq 0, \text{ and } \forall \vec{k} \in M \setminus \sigma^\vee, (\vec{v} + \vec{k}) \cdot \vec{v}_i < -a_i \text{ for some } i\}$ . Then  $P' = P_\sigma^D = \{\vec{v} \in M_\mathbb{R} \mid \forall i, 0 \leq \vec{v} \cdot \vec{v}_i < 1 - a_i\}$ .

*Proof.* We follow the proof of Lemma 3.11. Suppose that  $\vec{v} \in P_\sigma^D$ ,  $\vec{k} \in M$ , and  $\vec{k} \cdot \vec{v}_i < 0$ . Then  $(\vec{v} + \vec{k}) \cdot \vec{v}_i < (1 - a_i) + (-1) = -a_i$ , so  $\vec{v} + \vec{k} \notin \sigma'$ . It follows that  $P_\sigma^D \subset P'$ . On the other hand, suppose  $\vec{v} \notin P_\sigma^D$ . Either  $\vec{v} \cdot \vec{v}_i < 0$  for some  $i$ , in which case  $\vec{v} \notin P'$ , or  $\vec{v} \cdot \vec{v}_i \geq 1 - a_i$  for some  $i$ . In the latter case, we may, as in Lemma 3.11, choose  $\vec{k} \in M$  such that  $\vec{k} \cdot \vec{v}_i = -1$  and  $\vec{k} \cdot \vec{v}_j \geq 0$  for all  $j \neq i$ . Then  $\vec{k} \notin \sigma^\vee$ , but  $(\vec{v} + \vec{k}) \cdot \vec{v}_j \geq -a_j$  for all  $j$ . It follows that  $\vec{v} \notin P'$ .

We conclude that  $P' = P_\sigma^D$ .  $\square$

## 4.4 Triples Computation

**Definition 4.21.** Let  $\sigma$  be a cone as in Remark 3.1 and  $D$  a torus-invariant divisor, with corresponding polytope  $P_\sigma^D$  as in Definition 4.18. Let  $\mathfrak{a} \subset R$  be a monomial ideal, and  $0 \leq t \in \mathbb{R}$ . Let  $\text{Newt}(\mathfrak{a})$  denote the Newton polyhedron of  $\mathfrak{a}$ . We define  $P_\sigma^{\mathfrak{D}, \mathfrak{a}}$  to be the polytope  $(P_\sigma^D - t \cdot \text{Newt}(\mathfrak{a})) \cap \sigma^\vee$ .

**Theorem 4.22.** Let  $R$  be the coordinate ring of an affine toric variety, with conventions as in Remark 3.1. Let  $D$  be a torus-invariant divisor as in Definition 4.18. Let  $\mathfrak{a} \subset R$  be a monomial ideal, with associated polytope  $P_\sigma^{\mathfrak{D}, \mathfrak{a}}$  as in Definition 4.21. Then  $s(R, D, \mathfrak{a}^t) = \text{Volume}(P_\sigma^{\mathfrak{D}, \mathfrak{a}})$ .

*Proof of Theorem 4.22.* As in our proof of Lemma 4.19, we apply Lemma 4.8 to replace  $(p^e - 1)D$  by  $p^e D$ , and to assume that  $a_i \in \frac{1}{p^e} \mathbb{Z}$ . Likewise, we assume that  $t \in \frac{1}{p^e} \mathbb{Z}$ , and we replace  $(p^e - 1)t$  with  $p^e t$ , so that for sufficiently large  $e$ ,  $\lceil p^e t \rceil = p^e t$ , and  $\lceil p^e D \rceil = p^e D$ . We also replace  $\mathfrak{a}^{p^e t}$  with its integral closure  $\overline{\mathfrak{a}^{p^e t}}$ , which is generated by monomials in the set  $p^e t \cdot \text{Newt}(\mathfrak{a})$ .

We will use the characterization of  $F$ -signature of triples given in Lemma 4.9. Thus, we study  $(I_e^{\mathfrak{a}})^{1/p^e} = ((I_e^D)^{1/p^e} : (\overline{\mathfrak{a}^{p^e t}})^{1/p^e})$ . By Lemma 4.19,  $(I_e^D)^{1/p^e}$  is generated by the monomials whose characters lie in  $(\sigma^\vee \setminus P_\sigma^D) \cap \frac{1}{p^e} M$ . The set of characters  $\vec{v}$  with  $\chi^{\vec{v}} \in (\overline{\mathfrak{a}^{p^e t}})^{1/p^e}$  is  $(t \cdot \text{Newt}(\mathfrak{a})) \cap \frac{1}{p^e} M$ . Thus, the monomials in  $R^{1/p^e} \setminus (I_e^{\mathfrak{a}})^{1/p^e}$  are those  $\chi^{\vec{v}}, \vec{v} \in \frac{1}{p^e} M \cap \sigma^\vee$ , such that for some  $\vec{w} \in \frac{1}{p^e} M \cap t \cdot \text{Newt}(\mathfrak{a})$ ,  $\vec{v} + \vec{w} \in P_\sigma^D$ . This set of characters can be written as a Minkowski sum, so that the size  $a_e^{\mathfrak{a}}$  of the set is:

$$a_e^{\mathfrak{a}} = \#(((P_\sigma^D \cap \frac{1}{p^e} M) - ((t \cdot \text{Newt}(\mathfrak{a})) \cap \frac{1}{p^e} M)) \cap \sigma^\vee).$$

We obtain a slightly larger (but easier-to-count) set if we intersect with the lattice  $\frac{1}{p^e} M$  only after taking the Minkowski sum. In particular, set  $a'_e := \#((P_\sigma^D - t \cdot \text{Newt}(\mathfrak{a})) \cap \sigma^\vee \cap \frac{1}{p^e} M)$ . Note that  $a'_e = \#(P_\sigma^{\mathfrak{D}, \mathfrak{a}} \cap \frac{1}{p^e} M)$ . Now,  $a'_e$  may be larger than  $a_e^{\mathfrak{a}}$ . However, by Lemma 4.23,  $\lim_{e \rightarrow \infty} \frac{a_e^{\mathfrak{a}}}{p^{ed}} = \lim_{e \rightarrow \infty} \frac{a'_e}{p^{ed}}$ .

Thus,  $s(R, D, \mathfrak{a}^t) = \lim_{e \rightarrow \infty} \frac{a'_e}{p^{ed}}$ . We can apply Lemma 3.12 (with  $M = \mathbb{Z}^n$ ,  $P = P_\sigma^{\mathfrak{D}, \mathfrak{a}}$ , and  $a'_e = \#(P \cap \frac{1}{p^e} M)$ ) to conclude that the  $F$ -signature of triples is the volume of the polytope  $P_\sigma^{\mathfrak{D}, \mathfrak{a}}$ .  $\square$

## 4.5 A Technical Lemma

All that remains is to prove Lemma 4.23, which states that in the proof of Theorem 4.22, the quantities  $a_e^{\mathfrak{a}}$  and  $a'_e$  are “close enough” that either one may be used to compute  $F$ -signature. (These two quantities are obtained similarly: to compute  $a_e^{\mathfrak{a}}$ , we start with the polytopes  $P_\sigma^D$  and  $-t \cdot \text{Newt}(\mathfrak{a})$ ; intersect each with the lattice  $\frac{1}{p^e} M$ ; then take the Minkowski sum of these two sets. To compute  $a'_e$ , we take the Minkowski sum of the two polytopes, then intersect with the lattice  $\frac{1}{p^e} M$ .)

**Lemma 4.23.** Suppose we are in the situation of Theorem 4.22. Assume that all coefficients (the  $a_i$  and  $t$ ) lie in  $\frac{1}{p^{e_0}} M$  for some  $e_0$ . Let  $a_e^{\mathfrak{a}} = \#((\frac{1}{p^e} M \cap P_\sigma^D - \frac{1}{p^e} M \cap (t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D))$  and  $a'_e = \#((P_\sigma^D - (t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D) \cap \frac{1}{p^e} M \cap P_\sigma^D)$ . Then  $\lim_{e \rightarrow \infty} \frac{a'_e - a_e^{\mathfrak{a}}}{p^{ed}} = 0$ .

To prove Lemma 4.23, we wish to show that taking the Minkowski sum of two polytopes, then intersecting with a lattice, is “roughly the same” as performing those operations in reverse

order. That is the content of Lemma 4.29, the proof of which will proceed in several short steps. (Lemma 4.25 will be used to prove Lemma 4.26, and Lemmas 4.26, 4.27 and 4.28 will be used to prove Lemma 4.29. Lemmas 4.26 and 4.29 will then be used to prove Lemma 4.23.)

*Remark 4.24 (Notation).* In what follows, let  $M$  be a lattice, and  $M_{\mathbb{R}} = M \otimes \mathbb{R} \simeq \mathbb{R}^n$ . Fix  $e_0 > 0$ , and set  $M' = \frac{1}{p^{e_0}}M$ . We denote by  $d(\vec{v}, U)$  the distance from a point  $\vec{v}$  to a set  $U$ , given by  $d(\vec{v}, U) = \inf_{\vec{u} \in U} d(\vec{v}, \vec{u})$ . We denote by  $\mathcal{B}(\vec{v}, r)$  the ball of radius  $r$  around  $\vec{v}$ , given by  $\{\vec{u} \mid d(\vec{v}, \vec{u}) < r\}$ . We denote by  $\partial U$  the boundary of a set  $U$ . Set  $\lfloor x \rfloor_e = \frac{\lfloor p^e x \rfloor}{p^e}$  (rounding down to the nearest multiple of  $\frac{1}{p^e}$ ) and  $[x]_e = x - \lfloor x \rfloor_e$  (the  $\frac{1}{p^e}$ -fractional part of  $x$ ).

**Lemma 4.25.** *Fix any polytope  $Q \subset M_{\mathbb{R}}$  with extremal points in  $M'$ . There is a constant  $K$  such that for each  $e$ , and for  $\vec{u} \in Q$ ,  $d(\vec{u}, Q \cap \frac{1}{p^e}M) < \frac{K}{p^e}$ .*

*Proof.* Let  $\vec{u}_1, \dots, \vec{u}_k$  denote the extremal points of  $Q$ ; all are contained in  $\frac{1}{p^{e_0}}M$ . Fix  $\vec{v} \in Q$ , say  $\vec{v} = \sum_i a_i \vec{u}_i$  with  $0 \leq a_i$ ;  $\sum_i a_i = 1$ . Note that  $\sum_i [a_i]_e = 1 - \sum_i [a_i]_e \in \frac{1}{p^e}\mathbb{Z}$ . Furthermore,  $\sum_i [a_i]_e < \frac{k}{p^e}$  (each of the  $k$  terms in the sum is less than  $\frac{1}{p^e}$ ). Suppose without loss of generality that  $\vec{u}_1$  has the greatest length of any of the extremal points.

Set  $\vec{u}^* = \sum_i [a_i]_{e-e_0} \vec{u}_i + (\sum_i [a_i]_{e-e_0}) \vec{u}_1$ . Then  $\vec{u}^* \in Q \cap \frac{1}{p^e}M$ . Moreover,  $d(\vec{u}, \vec{u}^*) \leq \sum_i [a_i]_{e-e_0} |\vec{u}_i| + (\sum_i [a_i]_{e-e_0}) |\vec{u}_1| \leq \frac{2k}{p^{e-e_0}}$ . Thus, any  $K > 2kp^{e_0}$  will be sufficient.  $\square$

**Lemma 4.26.** *Fix polytopes  $Q, Q' \subset M_{\mathbb{R}}$  with extremal points in  $M'$ . For each  $e$ , set  $B_e = (Q_1 \cap \frac{1}{p^e}M) + (Q_2 \cap \frac{1}{p^e}M)$ ,  $B'_e = (Q_1 + Q_2) \cap \frac{1}{p^e}M$ . Then there is a constant  $K$  such that for each  $e$ , and for  $\vec{u} \in B'_e$ ,  $d(\vec{u}, B_e) < \frac{K}{p^e}$ .*

*Proof.* It suffices to show that given  $\vec{v} \in Q_1 + Q_2$ ,  $d(\vec{v}, Q_1 \cap \frac{1}{p^e}M + Q_2 \cap \frac{1}{p^e}M)$  is bounded above by some  $\frac{K}{p^e}$ . Fix such  $\vec{v}$ . Then  $\vec{v} = \vec{u} + \vec{w}$  with  $\vec{u} \in Q_1$  and  $\vec{w} \in Q_2$ . By Lemma 4.25, for some constants  $K_1, K_2$ , we have  $d(\vec{u}, Q_1 \cap \frac{1}{p^e}M) < \frac{K_1}{p^e}$ , and  $d(\vec{w}, (t \cdot Q_2) \cap \frac{1}{p^e}M) < \frac{K_2}{p^e}$ . It follows that  $d(\vec{v}, Q_1 \cap \frac{1}{p^e}M + Q_2 \cap \frac{1}{p^e}M) < \frac{K_1 + K_2}{p^e}$ .  $\square$

**Lemma 4.27.** *Let  $X$  be a polyhedron and  $\vec{v} \notin X$ . Then for some half-space  $H$ ,  $\vec{v} + H$  does not intersect  $X$ .*

*Proof.* Since  $\vec{v} \notin X$ ,  $\vec{v}$  lies in the complement,  $H$ , of some defining half-space of  $X$ . Thus,  $\vec{v} + H$  does not intersect  $X$ .  $\square$

**Lemma 4.28.** *Fix  $K > 0$ . There is a constant  $\kappa$  sufficiently large that for any half-space  $H \subset M_{\mathbb{R}}$ ,  $H \cap \mathcal{B}(\vec{0}, \kappa)$  contains an open ball of radius  $K$  around a lattice point  $\vec{w} \in H \cap M$ .*

*Proof.* Fix  $H$ . For some vector  $\vec{u}$ ,  $H = \{\vec{v} \in M_{\mathbb{R}} \mid \vec{v} \cdot \vec{u} \geq 0\}$ . We may assume  $|\vec{u}| = 1$ . Let  $\vec{e}_i$  be the standard basis vectors. Then for some  $i$ ,  $\frac{1}{n} \leq |\vec{u} \cdot \vec{e}_i|$ . If necessary, replace  $\vec{e}_i$  with  $-\vec{e}_i$  so that  $\vec{u} \cdot \vec{e}_i > 0$ . Set  $\kappa = \lceil nK \rceil + K$ . Then  $\vec{w} = \lceil nK \rceil \vec{e}_i$  satisfies the desired conditions:  $\vec{w} \in H \cap \mathcal{B}(\vec{0}, \kappa)$ ; since  $\vec{w} \cdot \vec{u} \geq K$ ,  $\mathcal{B}(\vec{w}, K) \subset H$ ; and clearly  $\mathcal{B}(\vec{w}, K) \subset \mathcal{B}(\vec{0}, \lceil nK \rceil + K)$ .  $\square$

**Lemma 4.29.** *Suppose we are in the situation of Lemma 4.26. Set  $P = Q_1 + Q_2$ . Then for some constant  $\kappa$ ,  $B'_e \setminus B_e$  is contained in an open neighborhood around the boundary  $\partial P$  of  $P$ , of radius  $\frac{\kappa}{p^e}$ . (That is, if  $\vec{v} \in B'_e \setminus B_e$ , then  $d(\vec{v}, \partial P) < \frac{\kappa}{p^e}$ .)*

*Proof.* Let  $K$  be a constant chosen as in Lemma 4.26 so that for all sufficiently large  $e$ , each monomial in  $B'_e$  lies within distance  $\frac{K}{p^e}$  of a point in  $B_e$ . Fix  $\kappa$  sufficiently large that for any half-space  $H$ ,  $H \cap \mathcal{B}(\vec{0}, \kappa)$  contains an open ball of radius  $K$  around a lattice point  $\vec{w} \in H \cap M$ . Suppose for the sake of contradiction that the claim is false. Then there is a monomial  $\vec{v} \in B'_e \setminus B_e$  such that  $\mathcal{B}(\vec{v}, \frac{\kappa}{p^e}) \subset P$ . Fix  $H$  so that  $\vec{v} + H$  does not intersect  $P$  (such  $H$  exists by Lemma 4.27). For our chosen  $K$ , fix  $\vec{w}$  as in Lemma 4.28.

Let  $\vec{u} = \vec{v} + \frac{1}{p^e} \vec{w}$ . Since  $d(\vec{v}, \vec{u}) < \frac{\kappa}{p^e}$ , we see that  $\vec{u} \in P$ , so  $\vec{u} \in B'_e$ . On the other hand, by our choice of  $\kappa$ ,  $\mathcal{B}(\vec{u}, \frac{K}{p^e}) \subset \vec{v} + H$ . It follows (by the statement of Lemma 4.28) that this ball does not intersect  $B_e$ , so  $d(\vec{u}, B_e) \geq \frac{K}{p^e}$ . Thus, by Lemma 4.26,  $\vec{u} \notin B'_e$ , a contradiction.  $\square$

*Proof.* (Proof of Lemma 4.23.) We apply Lemma 4.29 with  $Q_1 = P_\sigma^D$ ,  $Q_2 = t \cdot \text{Newt}(\mathfrak{a})$ , so that  $P_\sigma^{D, \mathfrak{a}} = (Q_1 + (-Q_2)) \cap \sigma^\vee$ . By Lemma 4.29, the difference  $a'_e - a_e^{\mathfrak{a}}$  is bounded by the number  $n_e$  of  $\frac{1}{p^e} M$ -lattice points in a neighborhood  $\mathcal{B}(\partial P_\sigma^{D, \mathfrak{a}}, \frac{\kappa}{p^e})$  of  $\partial P_\sigma^{D, \mathfrak{a}}$  of radius  $\frac{\kappa}{p^e}$ . The quantity  $\frac{n_e}{p^{ed}}$  is smaller than the volume of the union of all cubes intersecting  $\mathcal{B}(\partial P_\sigma^{D, \mathfrak{a}}, \frac{\kappa}{p^e})$ . That union of cubes is, in turn, contained in  $\mathcal{B}(\partial P_\sigma^{D, \mathfrak{a}}, \frac{\kappa + n}{p^{ed}})$ .

We conclude that  $\lim_{e \rightarrow \infty} \frac{n_e}{p^{ed}} \leq \lim_{e \rightarrow \infty} \text{Volume}(\mathcal{B}(\partial P_\sigma^{D, \mathfrak{a}}, \frac{\kappa + \sqrt{n}}{p^{ed}})) = 0$ , so  $\lim_{e \rightarrow \infty} \frac{n_e}{p^{ed}} = 0$ , as we desired to show.  $\square$

## 4.6 $\mathbb{Q}$ -Gorenstein Triples

Finally, we prove Corollary 4.30, which gives a particularly nice characterization of  $P_\sigma^{D, \mathfrak{a}}$  when  $(R, D)$  is a  $\mathbb{Q}$ -Gorenstein.

**Corollary 4.30.** *Let  $R$  be the coordinate ring of an affine toric variety,  $D$  a divisor on  $\text{Spec } R$ , and  $\mathfrak{a}$  a monomial ideal, presented as in Theorem 4.22. Suppose that the pair  $(R, D)$  is  $\mathbb{Q}$ -Gorenstein. Then  $s(R, D, \mathfrak{a}^t) = \text{Volume}(P_\sigma^D \cap t \cdot \text{Newt}(\mathfrak{a}))$ .*

*Proof of Corollary 4.30.* Since the pair  $(R, D)$  is  $\mathbb{Q}$ -Gorenstein, for some  $\vec{w} \in M \otimes \mathbb{Q}$ ,  $\vec{w} \cdot \vec{v}_i = 1 - a_i$  for each  $i$ . (Just let  $\vec{w}$  be the negative of the vector given by Lemma 4.17.) Set  $\phi$  to be the map  $\vec{u} \mapsto \vec{w} - \vec{u}$ . We claim that  $\phi$  is a volume-preserving bijection from  $P_\sigma^{D, \mathfrak{a}}$  to  $(t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D$ . The corollary will follow immediately.

Before we prove the claim, we first check that  $\phi$  maps  $P_\sigma^D$  to itself. Suppose  $\vec{z} \in P_\sigma^D$ . Then  $0 \leq \vec{z} \cdot \vec{v}_i$ , so  $(\vec{w} - \vec{z}) \cdot \vec{v}_i = (1 - a_i) - (\vec{z} \cdot \vec{v}_i) \leq 1 - a_i$ . Similarly,  $0 \leq (\vec{w} - \vec{z}) \cdot \vec{v}_i$ . We conclude that  $\phi(\vec{z}) \in P_\sigma^D$ .

Returning to our claim: the map  $\phi$  is clearly linear, volume-preserving, and self-inverse, so it suffices to show that  $\phi(P_\sigma^{D, \mathfrak{a}}) = (t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D$ . Suppose  $\vec{u} \in P_\sigma^{D, \mathfrak{a}}$ . In particular,  $\vec{u} \in P_\sigma^D$ , so (as we just showed)  $\phi(\vec{u}) \in P_\sigma^D$ .

Since  $\vec{u} \in P_\sigma^{D, \mathfrak{a}}$ , we may write  $\vec{u} = \vec{x} - \vec{y}$ , with  $\vec{x} \in P_\sigma^D$ ,  $\vec{y} \in (t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D$ . Then  $\vec{w} - \vec{u} = \vec{y} + (\vec{w} - \vec{x})$ . Since  $(\vec{w} - \vec{x}) \cdot \vec{v}_i \geq (1 - a_i) - (1 - a_i) = 0$ , we conclude that  $\vec{w} - \vec{x} \in \sigma^\vee$ . Since  $t \cdot \text{Newt}(\mathfrak{a})$  is closed under addition by vectors in  $\sigma^\vee$ , we conclude that  $\phi(\vec{u}) = \vec{y} + (\vec{w} - \vec{x}) \in t \cdot \text{Newt}(\mathfrak{a})$ .

So far, we've shown that  $\phi(P_\sigma^{D, \mathfrak{a}}) \subset P_\sigma^D \cap t \cdot \text{Newt}(\mathfrak{a})$ . On the other hand, suppose that  $\vec{y} \in P_\sigma^D \cap t \cdot \text{Newt}(\mathfrak{a})$ . We wish to show that  $\vec{w} - \vec{y} \in P_\sigma^{D, \mathfrak{a}}$ . Since  $\vec{w} \in P_\sigma^D$ ,  $\vec{w} - \vec{y} \in P_\sigma^D - ((t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D)$ . Moreover, since  $\vec{y} \in P_\sigma^D$ , we have that  $\phi(\vec{y}) \in P_\sigma^D$ . We conclude that  $\phi(\vec{y}) \in (P_\sigma^D - ((t \cdot \text{Newt}(\mathfrak{a})) \cap P_\sigma^D)) \cap P_\sigma^D = P_\sigma^{D, \mathfrak{a}}$ .

It follows that  $\phi$  is a volume-preserving bijection. Thus,  $s(R, D, \mathfrak{a}^t) = \text{Volume}(P_\sigma^{D, \mathfrak{a}}) = \text{Volume}(P_\sigma^D \cap t \cdot \text{Newt}(\mathfrak{a}))$ , as we desired to show.  $\square$

## 5 Alternative Monomial Ring Presentations

### 5.1 A Slightly More General $F$ -Signature Formula

Theorem 3.3 can be made to apply to monomial rings that are not quite presented “torically.” In particular, suppose  $R = k[S]$ , where  $S = L \cap \sigma^\vee$  for *any* lattice  $L$ , not just  $L = M$ . Then it’s not difficult to apply a slightly modified version of Theorem 3.3 to this presentation of  $R$ .

**Definition 5.1.** Let  $\sigma$  be a cone as in Remark 3.1, with primitive generators  $\vec{v}_1, \dots, \vec{v}_r$ . Let  $L$  be a lattice. For each  $i$ , let  $c_i = \min_{\vec{v} \in L} |\vec{v} \cdot \vec{v}_i|$ . We define  $P_\sigma^L \subset \sigma$  to be the polytope  $\{\vec{w} \in M_\mathbb{R} \mid \forall i, 0 \leq \vec{w} \cdot \vec{v}_i < c_i\}$ . (Note that if  $L = M$ , then  $P_\sigma^L = P_\sigma$ , as  $c_i = 1$  for each  $i$ .)

**Theorem 5.2.** (We use the conventions of Remark 3.1.) Let  $L \subset M$  be a sublattice, and set  $S = \sigma^\vee \cap L$ . (By Remark 2.11,  $L = \text{Lattice}(S)$ .) If  $\sigma$  is a full-dimensional cone, then  $s(R) = \text{Volume}(P_\sigma^L)$ , with the volume measured with respect to the lattice  $L$ . Moreover, for each  $e$ ,  $a_e = \#(P_\sigma^L \cap \frac{1}{p^e} L)$ .

*Proof.* The proof is essentially the same as that of Theorem 3.3 with  $M$  replaced by  $L$ , except that in the supporting Lemma 3.11, for each  $i$ , we replace the condition  $0 \leq \vec{v}_i < 1$  with  $0 \leq \vec{v}_i < c_i$ . (In the original proof, we made use of the fact that  $c_i = 1$  for  $L = M$ . It is easily checked that the proof holds in this more general case if we just replace each 1 with  $c_i$  as necessary.)  $\square$

*Example 5.3.* Theorem 5.2 may be used to recover the  $F$ -signature of a Veronese subring  $R^{(n)}$  of a polynomial ring  $R = k[x_1, \dots, x_n]$ . (This computation has already been performed in [HL02] and [Sin05].) In particular,  $R^{(n)} = k[\sigma^\vee \cap L]$ , where  $\sigma$  is the first orthant and  $L \subset M = \mathbb{Z}^n$  is the lattice of vectors whose coordinates sum to a multiple of  $n$ . It’s easily checked that for such  $L$  and  $\sigma$ ,  $c_i = 1$  for all  $i$ , so that  $P_\sigma^L = P_\sigma$ . Moreover,  $\#(M/L) = n$ , so  $\text{Volume}_L = \frac{1}{n} \cdot \text{Volume}_M$ . We conclude that  $s(R^{(n)}) = \text{Volume}_L(P_\sigma) = \frac{1}{n} \text{Volume}_M(P_\sigma) = \frac{1}{n} s(R)$ .

### 5.2 A New Proof of an Old $F$ -Signature Formula

Now we can provide an elementary proof of the  $F$ -signature formula given by Singh. First, we will need to discuss a few relevant properties of monomial rings.

**Definition 5.4.** Let  $S$  be a semigroup of monomials contained in the semigroup  $T$  generated by monomials  $x_1, \dots, x_n$ . (So  $k[A]$  is the polynomial ring  $k[x_1, \dots, x_n]$ .) Then  $S$  is full if  $\text{Frac } k[S] \cap k[x_1, \dots, x_n] = k[S]$ . Equivalently,  $\text{Lattice}(S) \cap T = S$ .

**Definition 5.5.** Let  $S$  be a semigroup of monomials contained in the semigroup  $T$  generated by monomials  $x_1, \dots, x_n$ . Then we say that  $S$  satisfies property  $(*)$  if the following holds: consider any variable  $x_i \in T$ . Then there exist monomials  $\zeta, \eta \in k[S]$  such that  $\frac{\zeta}{\eta}$ , as a fraction in  $\text{Frac } k[T]$  in lowest terms, can be written as  $\frac{\tau}{x_i}$  (where  $\tau$  is a monomial in  $S$  but not necessarily in  $T$ ). Equivalently, the lattice  $L \subset \mathbb{Z}^n$  generated by  $S$  should contain, for each  $i$ , an element with  $i^{\text{th}}$  coordinate equal to  $-1$ .

**Theorem 5.6** (Singh). Let  $R \subset A = k[x_1, \dots, x_n]$  be a subring generated by finitely many monomials,  $R = k[S]$ , where  $S$  is a finitely generated semigroup. Suppose  $k$  is perfect, and let  $m_A$  be the homogeneous maximal ideal of  $A$ . Assuming that  $R$  is presented so that  $S$  is full and satisfies property  $(*)$ , the  $F$ -signature of  $R$  is  $s(R) = \lim_{e \rightarrow \infty} \frac{l(R/(m_A^{[p^e]} \cap R))}{p^{ed}}$ . In particular,  $a_e = l(R/(m_A^{[p^e]} \cap R))$ .



*Proof.* We are given that  $S = \text{Lattice}(S) \cap T = \text{Lattice}(S) \cap \sigma^\vee$ , where  $\sigma$  is the first orthant, with primitive generators  $\vec{v}_i$  equal to the unit vectors in  $\mathbb{R}^n$ . Thus, we may apply Theorem 5.2 to the cone  $\sigma$  and the lattice  $L = \text{Lattice}(S)$ . It remains only to show that  $l(R/(m_A^{[p^e]} \cap R)) = \#(P_\sigma^L \cap \frac{1}{p^e}L)$ . Since  $c_i = 1$  for each  $i$ , the right-hand side is  $\#\{\vec{v} \in \frac{1}{p^e}L \mid 0 \leq \vec{v} \cdot \vec{v}_i < 1\}$ . The left-hand side is equal to the number of  $\vec{v} \in L$  whose coordinates are all less than  $p^e$ , which is  $\#\{\vec{v} \in L \mid 0 \leq \vec{v} \cdot \vec{v}_i < p^e\}$ . Dividing all elements of the left-hand side by  $q$ , we see that the left-hand side and right-hand side are equal. Thus,  $s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{ed}} = \lim_{e \rightarrow \infty} \frac{l(R/(m_A^{[p^e]} \cap R))}{p^{ed}}$ .  $\square$

## 6 Appendix

### 6.1 Some supporting $F$ -signature results

Remark 6.1 generalizes our main theorem to the case of an imperfect residue field. Theorem 6.2, computing the  $F$ -signature of a product, is used in our proof of theorem 3.3 as a means of avoiding discussion of the special case that our toric variety has torus factors. Theorem 6.4 is the local version of Theorem 6.2. Lemma 6.6 demonstrates the equivalence of local and  $\mathbb{N}$ -graded  $F$ -signature computations.

*Remark 6.1.* We wish to extend Theorem 3.3 to the case of an imperfect (but still  $F$ -finite) residue field. One can show using [Yao06] (Remark 2.3) that  $F$ -signature is residue field-independent. We give an less general but more concrete argument. Suppose  $k$  is not perfect. The arguments of Theorem 3.3 still compute the asymptotic growth rate of the number of splittings of  $k[\frac{1}{p^e}S]$ :

$$\lim_{e \rightarrow \infty} \frac{\text{free rank}(k[\frac{1}{p^e}S])}{p^{ed}} = \text{Volume}(P_\sigma).$$

But for imperfect  $k$ ,  $R^{1/p^e} = k^{1/p^e}[\frac{1}{p^e}S] \simeq k^{1/p^e} \otimes_k k[\frac{1}{p^e}S]$ . In particular,  $R^{1/p^e}$  is a free  $k[\frac{1}{p^e}S]$ -module of rank  $[k^{1/p^e} : k] = p^{e\alpha}$ . It follows that the free rank of  $R^{1/p^e}$  is  $p^{e\alpha}$  times the free rank of  $k[\frac{1}{p^e}S]$ . Thus, by Definition 2.4, as well as the above formula, we see immediately that  $s(R) = \text{Volume}(P_\sigma)$ .

This approach generalizes to the case of pairs and triples. In the pairs case, let  $\Sigma_e^D$  denote the set of monomials in  $R(p^e D)^{1/p^e}$ . Regardless of whether  $k$  is perfect, the arguments of Theorem 4.19 still compute the asymptotic growth rate of the number of splittings of  $k[\Sigma_e^D]$  that also split from  $k[\frac{1}{p^e}S]$  to be  $\text{Volume}(P_\sigma^D)$ . However,  $R(p^e D)^{1/p^e} \simeq k^{1/p^e} \otimes_k k[\Sigma_e^D]$ , so the number of splittings of  $R(p^e D)^{1/p^e}$  that also split from  $R^{1/p^e}$  is  $p^{e(d+\alpha)} \cdot \text{Volume}(P_\sigma^D)$ , and  $s(R, D) = \text{Volume}(P_\sigma^D)$ , as we desired to show.

In the triples case, let  $\Sigma_e^{D, \mathfrak{a}}$  denote the set of monomials in  $((I_e^D)^{1/p^e} : (\overline{\mathfrak{a}^{p^e t}})^{1/p^e})$ . Regardless of whether  $k$  is perfect, the arguments of Theorem 4.22 still compute the asymptotic growth rate of  $k[\frac{1}{p^e}S]/(\Sigma_e^{D, \mathfrak{a}})$  to be  $\text{Volume}(P_\sigma^{D, \mathfrak{a}})$ . Now consider Lemma 4.9. We see that  $F_*^e R/((I_e^D)^{1/p^e} : (\overline{\mathfrak{a}^{p^e t}})^{1/p^e}) \simeq k^{1/p^e} \otimes_k k[\frac{1}{p^e}S]/(\Sigma_e^{D, \mathfrak{a}})$ . Thus,  $a_e^{\mathfrak{a}} = p^{e(d+\alpha)} \cdot \text{Volume}(P_\sigma^{D, \mathfrak{a}})$ , and  $s(R, D, \mathfrak{a}) = \text{Volume}(P_\sigma^{D, \mathfrak{a}})$ , as we desired to show.

Now we'll show that the  $F$ -signature of a product of varieties (i.e., the  $F$ -signature a tensor product of rings over the appropriate field) is the product of the  $F$ -signatures. We'll give proofs in both the local and graded cases.

**Theorem 6.2.** *Let  $A$  be a semigroup (e.g.,  $\mathbb{N}$  or  $\mathbb{Z}^n$ ). Let  $R$  and  $S$  be  $A$ -graded rings containing a perfect field  $k$ , each with zeroth graded piece equal to  $k$ . Then  $s(R \otimes_k S) = s(R) \cdot s(S)$ .*

*Proof.* First, note that since  $k$  is perfect,  $(R \otimes_k S)^{1/p^e} \simeq R^{1/p^e} \otimes_k S^{1/p^e}$ . Suppose that  $R^{1/p^e} \simeq_{R\text{-mod}} R^{\oplus a_e} \oplus M_e$ , and  $S^{1/p^e} \simeq_{S\text{-mod}} S^{\oplus b_e} \oplus N_e$ . Then  $(R \otimes_k S)^{1/p^e} \simeq_{(R \otimes_k S)\text{-mod}} (R^{\oplus a_e} \oplus M_e) \otimes_k (S^{\oplus b_e} \oplus N_e) \simeq (R \otimes_k S)^{\oplus a_e b_e} \oplus (R \otimes_k N_e)^{\oplus a_e} \oplus (M_e \otimes_k S)^{\oplus b_e}$ . By Lemma 6.3,  $(R \otimes_k N_e)$  and  $(M_e \otimes_k S)$  have no free summands. It follows immediately that the free rank of  $(R \otimes_k S)^{1/p^e}$  is  $a_e b_e$ . Since  $\dim(R \otimes_k S) = \dim R + \dim S$ , we conclude that  $s(R \otimes_k S) = \lim_{e \rightarrow \infty} \frac{a_e b_e}{p^e(\dim R + \dim S)} = s(R) \cdot s(S)$ .  $\square$

**Lemma 6.3.** *Let  $A$  be a semigroup, and let  $R$  and  $S$  be  $A$ -graded rings over a field  $k$  with zeroeth graded piece equal to  $k$ . Let  $M$  and  $N$  be graded  $R$ - and  $S$ -modules, respectively. Suppose  $M \otimes_k N$  has a free summand as a graded  $R \otimes_k S$ -module. Then both  $M$  and  $N$  have free summands as graded  $R$ - and  $S$ -modules.*

*Proof.* Suppose that  $M \otimes_k N$  has a free summand. Then we have a graded map  $\phi : M \otimes_k N \rightarrow R \otimes_k S$ . In particular,  $1 \in \text{im } \phi$ . Since  $\text{im } \phi$  is generated by the images of graded simple tensors in  $M \otimes_k N$ , it follows that there is a graded simple tensor  $x \otimes y \in M \otimes_k N$  such that  $\deg \phi(x \otimes y) = 0$  but  $\phi(x \otimes y) \neq 0$ . The degree zero part of  $R \otimes_k S$  is isomorphic to  $k$ , so (after replacing  $\phi$  by  $\frac{1}{\phi(x \otimes y)} \phi$ ) we may assume that  $\phi(x \otimes y) = 1 \in R \otimes_k S$ .

Now, 1 generates a free summand  $R \otimes 1$  of the free  $R$ -module  $R \otimes_k S$ , so there is an  $R$ -module map  $\psi : R \otimes_k S \rightarrow R$  sending  $\phi(x \otimes y) \mapsto 1$ . Consider the map  $\psi \circ \phi : M \rightarrow R \otimes_k S \rightarrow R$ . This map sends  $x \mapsto \phi(x \otimes y) \mapsto 1$ . We conclude that  $M$  has a free summand as an  $R$ -module. By a symmetric argument,  $N$  has a free summand also.  $\square$

**Theorem 6.4.** *Suppose  $(R, m_R, k)$  and  $(S, m_S, k)$  are local rings containing the same perfect residue field  $k$ . Let  $m \subset R \otimes_k S$  be the maximal ideal  $m_R \otimes_k S + R \otimes_k m_S$ . Then  $s((R \otimes_k S)_m) = s(R) \cdot s(S)$ .*

*Proof.* First, note that since  $k$  is perfect,  $(R \otimes_k S)^{1/p^e} \simeq R^{1/p^e} \otimes_k S^{1/p^e}$ . Suppose that  $R^{1/p^e} \simeq_{R\text{-mod}} R^{\oplus a_e} \oplus M_e$ , and  $S^{1/p^e} \simeq_{S\text{-mod}} S^{\oplus b_e} \oplus N_e$ . Then  $(R \otimes_k S)^{1/p^e}_m \simeq_{(R \otimes_k S)_m\text{-mod}} ((R^{\oplus a_e} \oplus M_e) \otimes_k (S^{\oplus b_e} \oplus N_e))_m \simeq (R \otimes_k S)^{\oplus a_e b_e}_m \oplus (R \otimes_k N_e)^{\oplus a_e}_m \oplus (M_e \otimes_k S)^{\oplus b_e}_m$ . By Lemma 6.5,  $(R \otimes_k N_e)_m$  and  $(M_e \otimes_k S)_m$  have no free summands. It follows immediately that the free rank of  $(R \otimes_k S)^{1/p^e}_m$  is  $a_e b_e$ . Since  $\dim(R \otimes_k S)_m = \dim R + \dim S$ , we conclude that  $s((R \otimes_k S)_m) = \lim_{e \rightarrow \infty} \frac{a_e b_e}{p^e(\dim R + \dim S)} = s(R) \cdot s(S)$ .  $\square$

**Lemma 6.5.** *Let  $(R, m_R, k)$  and  $(S, m_S, k)$  be local rings with residue field  $k$ . Let  $m \subset R \otimes_k S$  be the maximal ideal  $m_R \otimes_k S + R \otimes_k m_S$ . Let  $M$  and  $N$  be  $R$ - and  $S$ -modules, respectively. Suppose  $(M \otimes_k N)_m$  has a free summand as an  $(R \otimes_k S)_m$ -module. Then both  $M$  and  $N$  have free summands as  $R$ - and  $S$ -modules.*

*Proof.* Suppose that  $(M \otimes_k N)_m$  has a free summand. Then we have a map  $\phi : (M \otimes_k N)_m \rightarrow (R \otimes_k S)_m$ . Since  $\phi \in \text{Hom}_{((R \otimes_k S)_m)}((M \otimes_k N)_m, (R \otimes_k S)_m) \simeq (\text{Hom}_{R \otimes_k S}(M \otimes_k N, R \otimes_k S))_m$ , by clearing denominators we may assume that  $\phi$  maps  $M \otimes_k N \rightarrow R \otimes_k S$  so that  $\text{im } \phi \not\subset m(R \otimes_k S)$ . It follows that there is a simple tensor  $x \otimes y \in M \otimes_k N$  such that  $\phi(x \otimes y) \notin m(R \otimes_k S)$ . In particular,  $\phi(x \otimes y) \notin m_R(R \otimes_k S)$ . Recall that if  $A$  is a free module over a Noetherian local ring  $(R, m_R)$ , then each element of  $A \setminus m_R A$  generates a free summand of  $A$ . Thus,  $\phi(x \otimes y)$  generates a free summand of the free  $R$ -module  $R \otimes_k S$ , and there is an  $R$ -module map  $\psi : R \otimes_k S \rightarrow R$  sending  $\phi(x \otimes y) \mapsto 1$ .

Now consider the map  $\psi \circ \phi : M \rightarrow R \otimes_k S \rightarrow R$ . This map sends  $x \mapsto \phi(x \otimes y) \mapsto 1$ . We conclude that  $M$  has a free summand as an  $R$ -module. By a symmetric argument,  $N$  has a free summand also.  $\square$

The following lemma (in particular, Lemma 6.6.5) demonstrates that if  $R$  is  $\mathbb{N}$ -graded with homogeneous maximal ideal  $m$ , then the (graded)  $F$ -signature of  $R$  and the (local)  $F$ -signature of  $R_m$  are equal. Moreover, the ideal  $I_e \subset R_m$  has a “graded counterpart”  $I_e^{gr} \subset R$  which may be used to define  $F$ -signature in the graded category.

**Lemma 6.6.** *Let  $R$  be an  $\mathbb{N}$ -graded ring over a field  $k$  and homogeneous maximal ideal  $m$ . Set  $I_e^{gr} = \{r \in R \mid \forall \phi \in \text{Hom}_R(R^{1/p^e}, R), \phi(r^{1/p^e}) \in m\}$ . Let  $I_e \subset R_m$  be the ideal defined in Definition 2.6, and  $i : R \rightarrow R_m$  the natural map. Then:*

1.  $F_*^e I_e^{gr}$  is the kernel of the map  $\psi : F_*^e R \rightarrow \text{Hom}_R(\text{Hom}_R(F_*^e R, R), R/mR)$  given by  $x \mapsto (\phi \mapsto \phi(x) + mR)$ .
2.  $I_e^{gr}$  is homogeneous.
3.  $I_e^{gr} = i^{-1}I_e = I_e \cap R$ .
4. If  $a_e(R)$  is defined as in 2.4, then  $a_e(R) = l(F_*^e R / F_*^e I_e^{gr})$ .
5. For each  $e$ ,  $a_e(R) = a_e(R_m)$ . Thus,  $s(R) = s(R_m)$ .

*Proof.* 1. This follows immediately from the definition.

2. Since  $F_*^e R$ ,  $R$ , and  $R/m$  are graded  $R$ -modules, so are the modules of homomorphisms between them. The map  $\psi$  is degree-preserving, that is, it's homogeneous of degree zero. It follows that  $F_*^e I_e^{gr} = \ker \psi$  is a graded submodule. Hence,  $I_e^{gr}$  is a homogeneous ideal.
3. Clearly  $F_*^e I_e = \ker(\psi \otimes_R R_m)$ . It follows that  $F_*^e I_e \simeq F_*^e I_e^{gr} \otimes_R R_m$  (by the flatness of localization). Thus,  $I_e = I_e^{gr} R_m = i(I_e^{gr})$ . Also,  $i : R \rightarrow R_m$  is injective. (If  $i(x) = 0$ , then  $w x = 0$  for  $w \notin m$ . If  $x_i$  is the lowest-degree term of  $x$ , and  $w_0$  is the degree-zero term of  $w$ , then  $w_0 x_i = 0$ , but  $w_0 \neq 0 \in k$ , so  $x_i = 0$ . Thus,  $x = 0$ .) We conclude that  $I_e^{gr} = I_e \cap R$ .
4. Suppose  $F_*^e R = R^{\oplus a_e} \oplus M_e$ , where the decomposition is graded and  $M_e$  has no graded free summands. We'll show that  $F_*^e I_e^{gr} = m R^{\oplus a_e} \oplus M_e$ , from which the claim follows immediately. Since  $I_e^{gr}$  is graded, to compute  $I_e^{gr}$  or  $F_*^e I_e^{gr}$ , we need only check which homogeneous elements they contain. Suppose  $x \in F_*^e I_e^{gr}$ . Then  $\phi(x) \notin m$  for some  $\phi$ . Without loss of generality, we may assume that  $\phi$  is homogeneous. Then  $\phi(x)$  is homogeneous and not in  $m$ , hence  $\phi(x) \in k$ , so  $x$  generates a direct summand  $Rx$  of  $F_*^e R$  that splits off from  $F_*^e R$ . This cannot occur if  $x \in m R^{\oplus a_e} \oplus M_e$ , so  $m R^{\oplus a_e} \oplus M_e \subset F_*^e I_e^{gr}$ . On the other hand, suppose  $x$  is homogeneous and  $x \notin m R^{\oplus a_e} \oplus M_e$ . This occurs precisely when  $x$  is a generator of one of the copies of  $R$  in the decomposition. In that case, clearly  $x \notin F_*^e I_e^{gr}$ . We conclude that  $F_*^e I_e^{gr} = m R^{\oplus a_e} \oplus M_e$ . The claim follows immediately.
5. It's easily checked that  $m^{[p^e]} R_m \subset I_e$ , so  $m^{[p^e]} R \subset I_e^{gr}$ . Thus,  $\text{Rad } I_e^{[gr]} = m$ , so  $R/I_e^{gr} = R_m/I_e^{gr} R_m$  as  $k$ -vector spaces. We conclude that  $l(R/I_e^{gr}) = l(R_m/I_e)$ . It follows that  $a_e(R) = l(F_* R / F_* I_e^{gr}) = l(F_* R_m / F_* I_e) = a_e(R_m)$ , as we desired to show.  $\square$

*Remark 6.7.* Lemma 6.6 doesn't actually apply to all monomial rings. In particular, a monomial ring arising from a cone  $\sigma$  that is not full-dimensional does not admit an  $\mathbb{N}$ -grading with zeroth graded piece  $k$ . On the other hand, we may apply Theorem 6.2 to reduce to the case where Lemma 6.6 applies. (If  $\sigma$  is not full-dimensional, then  $k[\sigma^\vee \cap M]$  decomposes as a tensor product of the coordinate ring of a torus, which has  $F$ -signature equal to 1, and a monomial ring arising from a full-dimensional cone, which does admit an  $\mathbb{N}$ -grading: see Lemma 2.9.)

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